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FACILITY FORM 602

N64-28747	
(ACCESSION NUMBER)	(THRU)
111	1
(PAGES)	(CODE)
CR-58540	33
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

OTS PRICE

XEROX $ 9.60 ph

MICROFILM $

REPUBLIC
AVIATION CORPORATION

S04-14803

RAC 406-4
(RD-817-454)
8 March 1963

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Summary Report

THE DETERMINATION OF THE NATURAL MODES OF VIBRATION FOR LARGE SYSTEMS

RAC 406-4

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Contract No. NAS 8-2646

Applied Research and Development
REPUBLIC AVIATION CORPORATION
Farmingdale, L. I., New York

FOREWORD

This final report was prepared by C. J. Meissner, Dr. R. S. Levy, S. Telles and J. H. Berman under Contract NAS-8-2646 between Republic Aviation Corporation, Farmingdale, New York and the Marshall Space Flight Center, NASA, Huntsville, Alabama. The contract covers the time period from 9 April 1962 through 9 December 1962.

The authors gratefully acknowledge the assistance of S. Pines, N. Levine, E. Leo and G. Rasanen.

ABSTRACT

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This report contains a method for obtaining the influence coefficients and natural modes of vibration for a complex structure.

In Part A the flexibility matrix method is described for obtaining the influence coefficients of a complex structure. The advantage of this method is that large structures may be analyzed without the necessity of inverting large matrices.

Part B describes Lanczos' method for obtaining eigenvalues and eigenvectors of large matrices. This method has the ability to extract the eigenvalues and associated eigenvectors even when the eigenvalues are extremely close together or are in fact multiple, as well as when the eigenvalues cover a large spread. The digital computer program for Part B on the IBM 7090 computer is included in this portion of the report.

Part C is a sample problem incorporating the methods described in Parts A and B.

Author

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PART A

STRUCTURAL ANALYSIS OF COMPLEX REDUNDANT STRUCTURES

SECTION I - INTRODUCTION

The structures of aerospace vehicles, of which the Saturn is a prime example, are often large and usually statically indeterminate. Those structures which can be idealized by an assemblage of one or two dimensional elements may be analysed by either the flexibility matrix method or the stiffness matrix method. Because of the large number of elements and the relatively small number of redundant forces in vehicles of the Saturn type, the flexibility matrix method is preferred. Also, the numerical problem of inverting large matrices is eliminated.

This report develops a method of obtaining the flexibility matrix of such structures by elastically coupling redundant component structures into a complex structure. Although this concept is not novel[1, 2], it is felt that the detailed ideas which make this method practical have not been sufficiently explained. Therefore, the method is rederived and particular attention is given to definitive statements regarding the nature and method of calculating the internal force influence matrices which are obtainable from equilibrium conditions, and which transform external unit applied loads or redundants into forces on the component structures or into internal forces on their structural elements. This includes axis transformation for elements in three dimensional space and an organized method that categorizes the various force influences, so that the force influence matrices that are the "coupling" matrices are easily understandable and calculable. This latter method is given in Section III, "Statically Indeterminate Coupling of Redundant Components of a Complex Structure." The practical application of the method makes the use of digital computers mandatory to perform the various matrix operations. The input data is in the form of the force influence matrices and flexibility matrices of standard structural elements. Novel idealizations are often possible which yield flexibility matrices that allow superior representations of the compatibility or equilibrium conditions where structure elements join. Therefore, the method is set up in such a way that any new element force and flexibility matrices can be used as they are calculated, without having to modify the basic digital program.

The redundant - internal force - and deflection influence coefficient matrices are derived, using the equality of internal and external work of deformation.

A procedure suggests itself which will permit the build up of the matrices of extremely complex structures from solutions of statically indeterminate structural subdivisions of reasonable complexity, requiring only the inversion of small matrices and various elementary matrix operations. The matrices involved are flexibility matrices of the simplest structural elements comprising the components, and internal force influence matrices.

SECTION II - FLEXIBILITY OF STRUCTURAL ELEMENTS

A. THE BEAM ELEMENT FLEXIBILITY MATRIX

The cantilevered beam element with two axes of cross-sectional symmetry is assumed to be the smallest basic element of that part of a structural network consisting of beams. The static deflection response of such an element to unit forces applied at its free end is expressed by its flexibility matrix, which is the matrix of coefficients $\left[\gamma_{\ell m}\right]$ of the generalized forces in the expression of the deflections:

$$\left[\gamma_{\ell m}\right]\left\{F_m\right\} = \left\{\Delta_m\right\} \tag{A. 1}$$

The calculation of the elements of $\left[\gamma_{\ell m}\right]$ is based both on the Principle of Virtual Work and the assumption that the internal stresses and strains are linearly dependent, on the basis of the engineering beam theory.

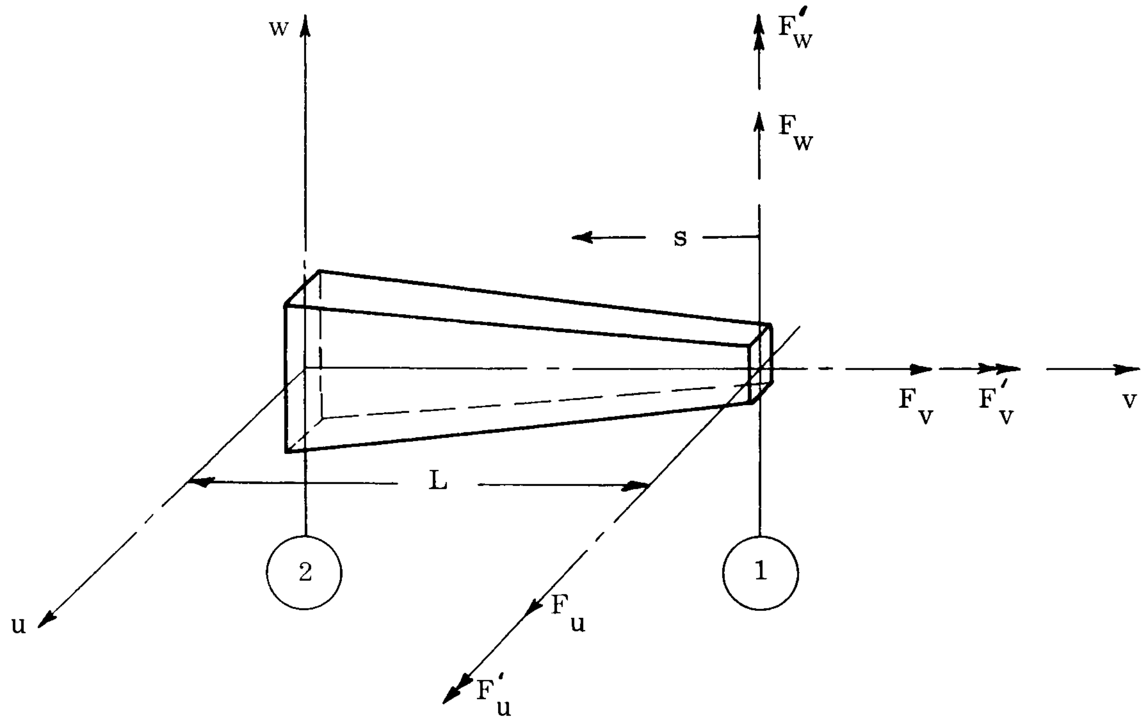


Figure A-1. Beam - Rod Element

Figure A-1 shows a beam element (1) - (2) fixed at its origin and free at point (1). The coordinate system coincides with the axes of symmetry. The six degrees of freedom of point (1) corresponds to the six components of $\{F_m\}$ shown in the figure. Thus

$$\{\Delta_m\} = \begin{Bmatrix} \Delta_u \\ \Delta_v \\ \Delta_w \\ \Delta'_u \\ \Delta'_v \\ \Delta'_w \end{Bmatrix}$$

and

$$\{F_m\} = \begin{Bmatrix} F_u \\ F_v \\ F_w \\ F'_u \\ F'_v \\ F'_w \end{Bmatrix}$$

where Δ = deflections

Δ' = rotations

F = forces

F' = moments

The cross-section properties at any point along the length of the beam are:

A_u = effective shear area loaded by F_u

A_v = effective axial area loaded by F_v

A_w = effective shear area loaded by F_w

I_u = moment of inertia about the u axis

I_v = torsional moment of inertia about the v axis

I_w = moment of inertia about the w axis

The non-zero elements of $\left[\gamma_{\ell m} \right]$ are:

$$\gamma_{uu} = \int_0^L \left(\frac{s^2}{EI_w} + \frac{1}{GA_u} \right) ds$$

$$\gamma_{vv} = \int_0^L \frac{ds}{EA_v}$$

$$\gamma_{ww} = \int_0^L \left(\frac{s^2}{EI_u} + \frac{1}{GA_w} \right) ds$$

$$\gamma_{u'w} = \gamma_{wu'} = \int_0^L \frac{sds}{EI_u}$$

$$\gamma_{u'u'} = \int_0^L \frac{ds}{EI_u}$$

$$\gamma_{v'v'} = \int_0^L \frac{ds}{GI_v}$$

$$\gamma_{w'u} = \gamma_{uw'} = -\int_0^L \frac{sds}{EI_w}$$

$$\gamma_{w'w'} = \int_0^L \frac{ds}{EI_w}$$

All other elements are zero.

Since practicality is of prime importance, it is recommended that the indicated integrations be performed by assuming that the elastic properties vary linearly between each pair of given numbers of point on each beam segment. The expanded form of Eq. (A. 1) is thus

$$
\begin{bmatrix}
\gamma_{uu} & & & & & \\
0 & \gamma_{vv} & & \text{(Symmetric about diagonal)} & & \\
0 & 0 & \gamma_{ww} & & & \\
0 & 0 & \gamma_{u'w} & \gamma_{u'u'} & & \\
0 & 0 & 0 & 0 & \gamma_{v'v'} & \\
\gamma_{w'u} & 0 & 0 & 0 & 0 & \gamma_{w'w'}
\end{bmatrix}
\begin{Bmatrix} F_u \\ F_v \\ F_w \\ F'_u \\ F'_v \\ F'_w \end{Bmatrix}
=
\begin{Bmatrix} \Delta_u \\ \Delta_v \\ \Delta_w \\ \Delta'_u \\ \Delta'_v \\ \Delta'_w \end{Bmatrix}
$$

B. THE ROD ELEMENT FLEXIBILITY MATRIX

A rod has only one degree of freedom, that of elongation of one end with respect to the other. If the beam element of Figure (A-1) is considered with only that degree of freedom, i. e., Δ_v , then the flexibility of the rod is γ_{vv}. Thus

$$\gamma_{vv} \; F_v = \Delta_v \tag{A.2}$$

where $\gamma_{vv} = \int_0^L \frac{ds}{EA_v}$

However, in the case of interaction of rods and shear panels, it is important to include the deflection of the rod due to unit value of an applied constant shear flow (Figure A-2).

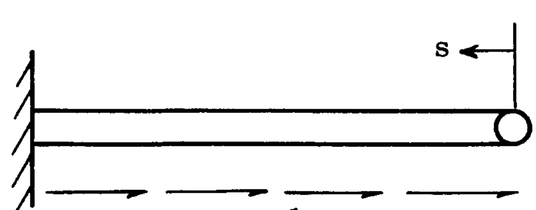


Figure A-2. Rod with Constant Shear Flow

The value of the deflection due to unit shear flow, where the subscript v″ applies to the shear flow, is

$$\gamma_{vv''} = \int_0^L \frac{s\,ds}{EA_v} \tag{A. 3}$$

Maxwell's Law of Reciprocity requires the existence of the term $\gamma_{v''v} = \gamma_{vv''}$. This is a generalized deformation due to the application of a unit end load, or otherwise interpreted, it is the work done by the unit shear flow as it displaces through deformations caused by the unit end load.

The corresponding diagonal term, $\gamma_{v''v''}$, is obtained by considering the applied load to be a unit shear flow and calculating the virtual work caused by that load.

Then

$$\gamma_{v''v''} = \int_0^L \frac{s^2 ds}{EA_v} \tag{A.4}$$

This is the work done by the unit shear flow as it moves through the deformation caused by it.

Thus

$$\left[\gamma\right]_{rod} = \begin{bmatrix} \gamma_{ii} & \gamma_{ii''} \\ \gamma_{i''i} & \gamma_{i''i''} \end{bmatrix} \tag{A.5}$$

where

$i = u, v \text{ or } w$

C. THE SHEAR PANEL FLEXIBILITY MATRIX

Some two dimensional elements in the form of skin panels or beam webs can be assumed to carry only pure constant shear flow (Figure A-3).

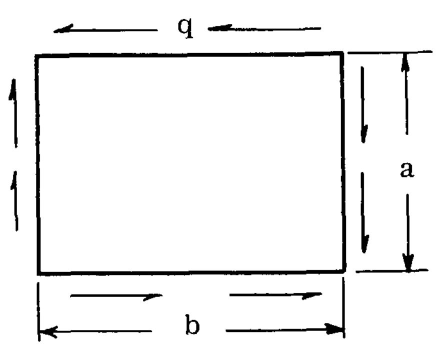


Figure A-3. Shear Panel

The shearing deformation for a unit shear flow is given by:

$$\gamma_{qq} = \frac{ab}{Gt} \tag{A.6}$$

If the panel has a rhomboid shape (Figure A-4), the skewed coordinates demand adjustment in this value according to the theory of elasticity.

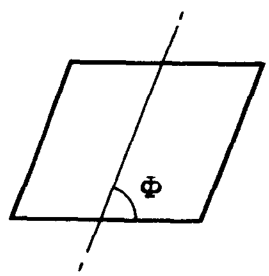


Figure A-4. Rhomboid Shear Panel

The flexibility is thereby increased and the value of the shearing deformation is then

$$\gamma_{qq} = \frac{ab}{Gt}\left(1 + \frac{G}{E}\cot^2\Phi\right) \tag{A.7}$$

where E = Young's modulus

G = shear modulus

Φ = angle of midline with respect to the side of the panel

The complementary strain energy is thus

$$U^* = \frac{1}{2}\gamma_{qq}\, q^2$$

D. FLEXIBILITY MATRICES OF ELEMENTS OF OTHER SHAPES

The flexibilities of rod and beam elements with various simple variations of cross section properties are given in Reference 3. The trapezoidal quadrilateral shear panel flexibilities are usually approximated by assuming equilibrium to be satisfied by suitably adjusted uniform shear flows on the edges.

Some of these approximations also appear in Reference 3. However, if a trapezoid is swept, similar to a rhomboid shape shear panel, the effect of this sweep must be incorporated by increasing the flexibilities, through the factor C, involving E, G and the average angle of sweep Φ:

$$C = \left(1 + 4\frac{G}{E}\cot^2\Phi\right)$$

SECTION III - STATICALLY INDETERMINATE COUPLING OF REDUNDANT COMPONENTS OF A COMPLEX STRUCTURE

In previous consideration of the analysis of statically indeterminate structures (3) internal forces were calculated which were caused by externally applied forces and redundants, respectively, on the cut (and thus statically determinate) structure. The concepts used are extended to the calculation of influence coefficients of complex structures consisting of statically indeterminate component structures, coupled so that redundant forces exist at the boundaries between the component structures.

A. ANALYSIS OF COMPONENT STRUCTURES

1. Basic Concepts for Solution of Component Structures

The structure is assumed to consist of interconnected elements in which the internal forces are statically determinate when sufficient cuts are made to remove redundancies (Figure A-5). Their individual idealizations permit the calculation of flexibility influence coefficients. These influence coefficients are used to calculate the deformations of the elements caused by applied loads. Linear structural behavior is assumed, permitting application of the superposition principle.

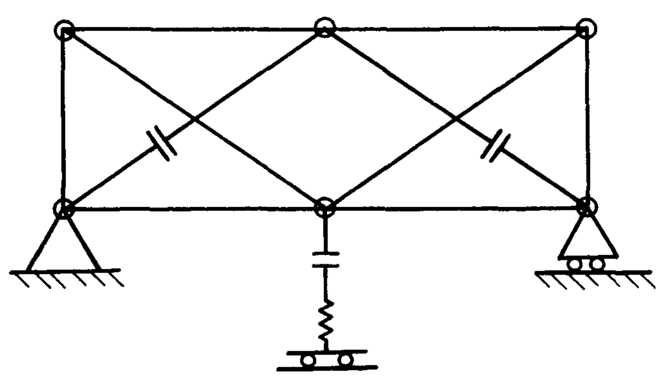


Figure A-5. Removal of Redundancies Through the Use of Cuts

Each element admits only a limited number of forces and corresponding kinematic motions. Examples are the slope and deflection of one end of a cantilever with respect to the fixed end caused by unit moments and shears. The deformation of the elements, with respect to their individual datums, are easily calculated by multiplying the matrices of the influence coefficients $[\gamma]$ of each of the elements by the force vector representing the forces $\{F\}$ sustained by it. These forces are due to external loads acting on the element or the internal forces due to load transfer between elements in the cut structure. The latter (e.g., shears and bending moments in a beam), can be determined of course from equations of static equilibrium of the structure that has been cut at all points of redundancy so that these internal forces are expressible in terms of the applied and redundant forces. The contributions to the internal forces from these two sources are expressed separately by the matrices $[\alpha_{mh}]$ and $[\beta_{mq}]$. The elements $[\alpha_{mh}]$ are the internal forces at m due to a unit load applied at h. Thus, when $[\alpha_{mh}]$ is multiplied by the external forces $\{P_h\}$, the resulting values are the internal forces $\{F_{mh}\}$ at interconnection points or loading points m. Thus,

$$\left\{F_{mh}\right\} = \left[\alpha_{mh}\right]\left\{P_h\right\} \tag{A.8}$$

Similarly, there are the internal forces $\{F_{mq}\}$ caused by the redundants $\{X_q\}$. The columns of the matrix $[\beta_{mq}]$ are the internal forces at m due to unit values of the redundants $\{X_q\}$ at q. Therefore,

$$\left\{F_{mq}\right\} = \left[\beta_{mq}\right]\left\{X_q\right\} \tag{A.9}$$

The total internal force is

$$\left\{F_m\right\} = \left\{F_{mh}\right\} + \left\{F_{mq}\right\} \tag{A.10}$$

or

$$\left\{F_m\right\} = \left[\alpha_{mh} \mid \beta_{mq}\right] \left\{\frac{P_h}{X_q}\right\} \tag{A.11}$$

A propped cantilever beam, loaded at two points, is shown in Figure A-6. The beam is broken up into two elements and a cut is made between the beam and the flexible support. The internal shears v_1 and v_2, the moment m_2, and the support force p are the forces transformed through α_{mh} and β_{mq} shown below.

$$\{F_m\} = \begin{Bmatrix} v_1 \\ v_2 \\ m_2 \\ p_1 \end{Bmatrix} = \left[\alpha_{mh} \;\middle|\; \beta_{mq} \right] \begin{Bmatrix} P_h \\ \hline X_q \end{Bmatrix}$$

$$= \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 1 & 1 & -1 \\ \ell_1 & 0 & -\ell_1 \\ 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} P_1 \\ P_2 \\ \hline X_1 \end{Bmatrix}$$

Thus

$$\left[\alpha_{mh}\right] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \ell_1 & 0 \\ 0 & 0 \end{bmatrix} ; \left[\beta_{mq}\right] = \begin{bmatrix} -1 \\ -1 \\ -\ell_1 \\ 1 \end{bmatrix}$$

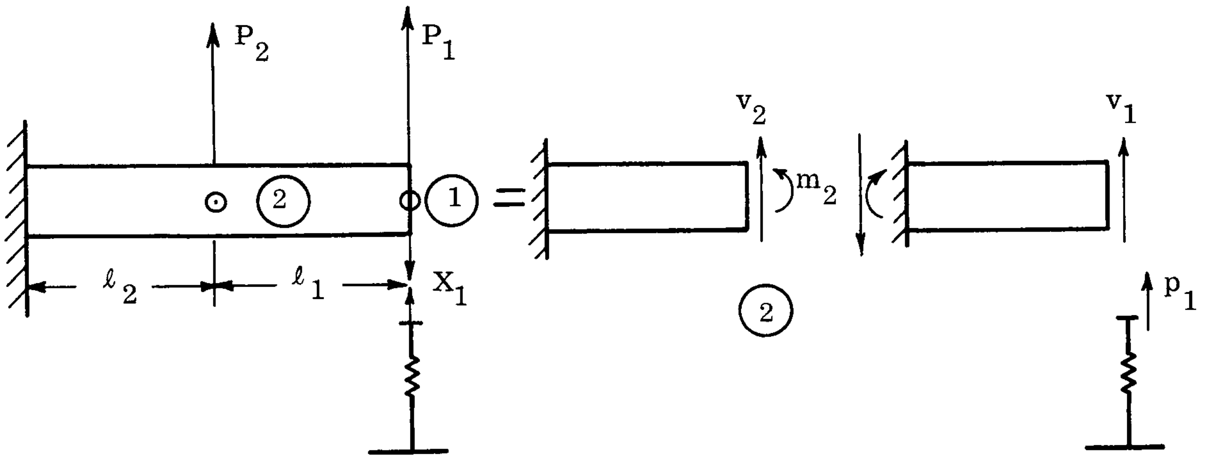


Figure A-6. Internal Forces in a Redundant Beam

Methods for finding $\{X_q\}$, and thus $\{F_m\}$, will now be obtained by considering the equality of the internal and external work of the structural deformations.

2. Internal Work

The internal work of the structure is obtained by summing the contributions of each element to that work. Only the work of the internal and external forces in and on an element relative to the element datum will be used to find the internal work. Actually, the absolute displacements, which are the sum of the datum and the relative displacements, could be used to find the work but the contribution of rigid body motion is zero because the complete force system on each element, including the reactions at its datum, is in equilibrium and the work required for a rigid body displacement of a system of forces in equilibrium is equal to zero.

The deflections of an element relative to its datum are

$$\left\{\Delta_{\ell}\right\} = \left[\gamma_{\ell m}\right] \left\{F_m\right\} \tag{A.12}$$

It is to be understood that, physically, ℓ and m may represent the same or different degrees of freedom; ℓ can be a deflection or a rotation type deformation and m can be a linear or a moment type force.

The structure is assumed to behave linearly and therefore the internal work W_{el}, the product of these forces and the deflections, is given by

$$W_{el} = \frac{1}{2} \left\{F_m\right\}^T \left\{\Delta_{\ell}\right\} \tag{A.13}$$

Substituting Eqs. (A.11) and (A.12) into (A.13)

$$W_{el} = \frac{1}{2} \left[\left[P_h\right]^T \middle| \left[X_q\right]^T \right] \begin{bmatrix} [\alpha_{mh}]^T \\ \hline [\beta_{mq}]^T \end{bmatrix}_{(el)} \left[\gamma_{\ell m}\right]_{(el)} \left[[\alpha_{mh}] \middle| [\beta_{mq}] \right]_{(el)} \left\{ \begin{array}{c} P_h \\ \hline X_q \end{array} \right\} \tag{A.14}$$

The total internal work in the structure is the sum of the work contributions of the elements:

$$W_{int} = \sum_{j=1}^{n} W_{el_j} \tag{A.15}$$

where

n = number of elements

Now $\left\{\frac{P_h}{X_q}\right\}$ will be common to all expressions of internal work for all structural elements such as Eq. (A. 14), provided $[\alpha_{mh}]$ and $[\beta_{mq}]$ are organized to have columns corresponding to every P_h and X_q sustained by the structure, even if some of these forces have zero influence. The sum of the internal works, Eq. (A. 15), is obtained by providing enough rows in $\left[[\alpha_{mh}] \,\vdots\, [\beta_{mq}]\right]$ and $[\gamma_{\ell m}]$ so that all internal forces in the component structure and their flexibilities are represented.

The deflections of a structure consisting of two elements would be:

$$\{\Delta_\ell\}_i = \left[\gamma_{\ell m}\right]_i \left[[\alpha_{mh}] \,\vdots\, [\beta_{mq}]\right]_i \left\{\frac{P_h}{X_q}\right\}$$

$$\{\Delta_\ell\}_{i+1} = \left[\gamma_{\ell m}\right]_{i+1} \left[[\alpha_{mh}] \,\vdots\, [\beta_{mq}]\right]_{i+1} \left\{\frac{P_h}{X_q}\right\}$$

or

$$\left\{\begin{array}{c}\{\Delta_\ell\}_i \\ \hline \{\Delta_\ell\}_{i+1}\end{array}\right\} = \left[\begin{array}{c|c}[\gamma_{\ell m}]_i & [0] \\ \hline [0] & [\gamma_{\ell m}]_{i+1}\end{array}\right]\left[\begin{array}{c|c}[\alpha_{mh}]_i & [\beta_{mq}]_i \\ \hline [\alpha_{mh}]_{i+1} & [\beta_{mq}]_{i+1}\end{array}\right]\left\{\begin{array}{c}P_h \\ \hline X_q\end{array}\right\} \quad \text{(A. 16)}$$

Note that the elements in the i^{th} set of rows of $[\alpha_{mh}]$ must therefore be the values of the internal forces in the i^{th} element due to unit values of external loads $\{P_h\}$ applied anywhere on the cut, statically determinate structure. The elements in the i^{th} set of rows of $[\beta_{mq}]$ are similarly the internal forces in the i^{th} element due to unit values of the redundants $\{X_q\}$.

The internal work is one half the product of the internal forces and the deflections. This work in two elements is:

$$\begin{aligned} W_{int} &= W_{el_i} + W_{el_{i+1}} \\ &= \frac{1}{2}\left[F_{m_i}^T \Delta_{m_i} + F_{m_{i+1}}^T \Delta_{m_{i+1}}\right] \qquad \text{(A. 17)} \\ &= \frac{1}{2}\left[\{F_{m_i}\}^T \,\vdots\, \{F_{m_{i+1}}\}^T\right]\left\{\begin{array}{c}\Delta_{m_i} \\ \hline \Delta_{m_{i+1}}\end{array}\right\} \end{aligned}$$

But

$$\left[\{F_{m_i}\}^T \,\middle|\, \{F_{m_{i+1}}\}^T \right] = \left[\{P_h\}^T \,\middle|\, \{X_q\}^T \right] \left[\begin{array}{c|c} [\alpha_{mh}]_i^T & [\alpha_{mh}]_{i+1}^T \\ \hline [\beta_{mq}]_i^T & [\beta_{mq}]_{i+1}^T \end{array} \right] \tag{A. 18}$$

Thus, for two elements,

$$W_{int} = \frac{1}{2} \left[P_h^T \,\middle|\, X_q^T \right] \left[\begin{array}{c|c} \alpha_{mh_1}^T & \alpha_{mh_2}^T \\ \hline \beta_{mq_1}^T & \beta_{mq_2}^T \end{array} \right] \left[\begin{array}{c|c} \gamma_{\ell m_1} & 0 \\ \hline 0 & \gamma_{\ell m_2} \end{array} \right] \left[\begin{array}{c|c} \alpha_{mh_1} & \beta_{mq_1} \\ \hline \alpha_{mh_2} & \beta_{mq_2} \end{array} \right] \left\{ \begin{array}{c} P_h \\ \hline X_q \end{array} \right\} \tag{A. 19}$$

in which various brackets and braces of the notation in (A. 18) have been deleted for clarity. Each of the letter symbols now stands for the corresponding matrix.

Generalizing this to any number of elements, the total internal work is:

$$W_{int} = \frac{1}{2} \left[P_h^T \,\middle|\, X_q^T \right] \left[\begin{array}{c} \alpha_{mh}^T \\ \hline \beta_{mq}^T \end{array} \right] \left[\diagdown \gamma_{\ell m} \diagdown \right] \left[\alpha_{mh} \,\middle|\, \beta_{mq} \right] \left\{ \begin{array}{c} P_h \\ \hline X_q \end{array} \right\} \tag{A. 20}$$

which has the same form as Eq. (A. 14), but gives the generalized expression of internal work for the whole structure through the foregoing definitions of $[\alpha_{mh}]$ and $[\beta_{mq}]$ and by designating $[\gamma_{\ell m}]$ to be a square, symmetrical matrix whose elements are uncoupled flexibility matrices of the elements along the diagonal, such as that in Eq. (A. 19) above.

3. External Work

The external work is expressed by summing the work of the external forces as they move through their displacements. It will be helpful to use the trick of "adding zero" in the derivation of the expression of the equality of internal and external work, so that terms involving the redundants in the accompanying matric equations may be understood.

The concept of an external force can be generalized to include the redundants which are applied to each side of a "cut" face, equal and opposite to each other. Compatibility of the cut faces requires that each side of the cut face move through the same absolute displacement. The external work of the redundants is therefore equal to zero because they are each equal and opposite forces, representing a zero vector, moving through some displacement with respect to an absolute datum. Addition of the "external work" of the redundants of the total external work is therefore adding zero.

The cut points and others which are loaded with external known forces also move because of the influence of the loads on them and the forcing of compatibility by the redundants. Then if the structure is cut so that all forces within it are statically determinate with respect to externally applied forces and the redundants, the resulting deflections are expressed through means of a flexibility matrix [c] referred to some common absolute datum for the structure. It is desired to obtain the displacements at the externally loaded points with respect to this datum. They are calculated from an equation, such as

$$\left\{\Delta_g\right\} = \left[c_{gh} \;\middle|\; c_{gq}\right]\left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \qquad \text{(A. 21)}$$

in which $\{\Delta_g\}$ are the deflections of the externally loaded points, g, and [c] is the flexibility matrix of the cut, statically determinate structure. The subscripts g and h pertain to applied loads or corresponding degrees of freedom at their points of application, and p and q pertain to redundants or the corresponding degrees of freedom at their points of application. Thus the partitions c_{gh} and c_{gq} are the g deformations due to unit values of forces P_h and X_q, respectively.

Consider the displacements $\left\{\Delta_p\right\}_{(a)}$ of one of the two faces of each cut. They can be similarly expressed as:

$$\left\{\Delta_p\right\}_{(a)} = \left[c'_{ph} \;\middle|\; c'_{pq}\right]\left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \qquad \text{(a)}$$

for the faces (a). For the faces (b) they would be:

$$\left\{\Delta_p\right\}_{(b)} = \left[c''_{ph} \;\middle|\; c''_{pq}\right]\left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \qquad \text{(b)}$$

The relative deformations of (a) and (b) are the differences. Thus

$$\left\{\Delta_p\right\} = \left\{\Delta_p\right\}_{(a)} - \left\{\Delta_p\right\}_{(b)}$$

The relative displacements of the cut faces can be expressed, as was done for the load point displacements, as

$$\left\{\Delta_p\right\} = \left[c_{ph} \mid c_{pq}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \tag{A. 22}$$

in which

$$\left[c_{ph}\right] = \left[c'_{ph}\right] - \left[c''_{ph}\right]$$

and

$$\left[c_{pq}\right] = \left[c'_{pq}\right] - \left[c''_{pq}\right]$$

which shows the use of the primed and double primed matrices of Eqs. (a) and (b).

The total work of the forces is given by the sum of the products of the forces and their displacements. Remembering that $\{\Delta_p\}$ is defined as the relative displacement vector of the cut faces, then the total external work (if $\{\Delta_p\}$ were not equal to zero) is

$$W_{ext} = \frac{1}{2}\left[\left\{P_h\right\}^T \left\{\Delta_g\right\} + \left\{X_q\right\}^T \left\{\Delta_p\right\}\right]$$

or

$$W_{ext} = \frac{1}{2}\left[\left\{P_h\right\}^T \mid \left\{X_q\right\}^T\right] \left\{\begin{array}{c} \Delta_g \\ \hline \Delta_p \end{array}\right\} \tag{A. 23}$$

Substituting Eq. (A. 21) and (A. 22) in Eq. (A. 23) gives

$$W_{ext} = \frac{1}{2}\left[P_h^T \mid X_q^T\right] \left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} \tag{A. 24}$$

Now, recalling the expression of internal work, Eq. (A.20), and setting

$$W_{ext} = W_{int}$$

there results, as shown in Appendix A:

$$\begin{bmatrix} \alpha_{mh}^{T} \\ \hline \beta_{mq}^{T} \end{bmatrix} \begin{bmatrix} \diagdown & & \\ & \gamma_{\ell m} & \\ & & \diagdown \end{bmatrix} \left[\alpha_{mh} \mid \beta_{mq} \right] = \left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array} \right] \qquad \text{(A. 25)}$$

A means of calculating $[\, c \,]$ has thus been found. The values of the redundants and the deformations of the compatible structure remain to be found. The internal forces can be obtained from Eq. (A. 11) once the redundants are known.

It can be seen from Eq. (A. 21) that if all the values of $\{ X_q \}$ for every separate application of a unit external load at point h were known, then the values of $\{ \Delta_g \}$ for any such unit load would form, by definition, one column of the external flexibility influence coefficient matrix, i.e., that the resulting deflections are caused by a unit force applied to the structure at h, and in which compatibility is enforced by corresponding values of $\{ X_q \}$.

To find these values of $\{ X_q \}$, the relative deformations $\{ \Delta_p \}$ in Eq. (A. 22) are set equal to zero, i.e., for compatibility of the cuts:

$$\left\{ \Delta_p \right\} = 0$$

Then

$$\left[c_{ph} \right] \left\{ P_h \right\} + \left[c_{pq} \right] \left\{ X_q \right\} = 0$$

Therefore

$$\left\{ X_q \right\} = -\left[c_{pq} \right]^{-1} \left[c_{ph} \right] \left\{ P_h \right\} \qquad \text{(A. 26)}$$

Substituting this result in Eq. (A. 21), gives the deflections of points (g) as:

$$\left\{ \Delta_g \right\} = \left[c_{gh} \mid c_{gq} \right] \begin{Bmatrix} P_h \\ \hline X_q \end{Bmatrix} \qquad \text{(A. 21)}$$

$$= \left[c_{gh} \mid c_{gq} \right] \begin{Bmatrix} P_h \\ \hline - c_{pq}^{-1} \; c_{ph} \, P_h \end{Bmatrix}$$

Thus

$$\left\{\Delta_g\right\} = \left[c_{gh} - c_{gq}\, c_{pq}^{-1}\, c_{ph}\right] \left\{P_h\right\} \qquad \text{(A. 27)}$$

Let

$$\left[\gamma_{gh}\right] = \left[c_{gh} - c_{gq}\, c_{pq}^{-1}\, c_{ph}\right] \qquad \text{(A. 28)}$$

Then

$$\left\{\Delta_g\right\} = \left[\gamma_{gh}\right] \left\{P_h\right\} \qquad \text{(A. 27a)}$$

which shows that $[\gamma_{gh}]$, according to definition is the flexibility influence coefficient matrix of the redundant structure with respect to its own datum.

The internal forces $\{F_m\}$ can be calculated for unit values of the applied forces $\{P_h\}$. The result is the internal force influence matrix $[f_{mh}]$. Substituting $\{X_q\}$ of Eq. (A. 26) into Eq. (A. 11),

$$\left\{F_m\right\} = \left[\alpha_{mh} \mid \beta_{mq}\right] \left\{\begin{array}{c} P_h \\ \hline -c_{pq}^{-1}\, c_{ph}\, P_h \end{array}\right\}$$

Thus

$$\left\{F_m\right\} = \left[\alpha_{mh} - \beta_{mq}\, c_{pq}^{-1}\, c_{ph}\right] \left\{P_h\right\} \qquad \text{(A. 29)}$$

Let

$$\left[f_{mh}\right] = \left[\alpha_{mh} - \beta_{mq}\, c_{pq}^{-1}\, c_{ph}\right]$$

Then

$$\left\{F_m\right\} = \left[f_{mh}\right] \left\{P_h\right\} \qquad \text{(A. 29a)}$$

which defines $[f_{mh}]$ as the internal force influence matrix of the uncut redundant structure.

4. Summary

Due to loads at points (h):

The redundants are

$$\left\{X_q\right\} = -\left[c_{pq}\right]^{-1} \left[c_{ph}\right] \left\{P_h\right\} \qquad \text{(A. 30)}$$

The internal forces are

$$\left\{F_m\right\} = \left[\alpha_{mh} - \beta_{mq}\, c_{pq}^{-1}\, c_{ph}\right] \left\{P_h\right\} \qquad \text{(A. 31)}$$

The deflections are

$$\left\{\Delta_g\right\} = \left[c_{gh} - c_{gq}\ c_{pq}^{-1}\ c_{ph}\right] \left\{P_h\right\} \tag{A.32}$$

It should be kept in mind that for certain structures consisting of many elements joining in few points it may be advantageous to obtain $[\gamma_{gh}]$ by inverting the stiffness matrix of the structure, which in such cases may be more easily obtainable (References 4 and 5).

B. ANALYSIS OF A STRUCTURAL COMPLEX

The method of solving statically indeterminate structures has been summarized previously in Eqs. (A.30) through (A.32). The method can now be extended to cover a class of structures that consist of statically indeterminate modules interconnected in a statically indeterminate way. It is proposed that through such a concept any extremely large structural complex can be solved by analyzing the properties of component structures, chosen at convenience, which may themselves be statically indeterminate, and coupling them through the satisfaction of equilibrium and compatibility conditions at their points of physical interconnection. The generalized relationships which lead to the expression of the interconnecting matrices will be developed, and as will be seen, the concepts that were used to develop Eqs. (A.30) through (A.32) are also applicable here. Furthermore, although this may be an extreme generalization, it can be seen that the properties of the statically indeterminate component structures may themselves have been obtained from a further breakdown into statically indeterminate sub-modules and a subsequent synthesis of the component structure through the compatible interconnection of these sub-modules. This can be continued ad infinitum, conceptually producing the picture-within-a-picture-within-a-picture effect.

The method will be developed assuming statically indeterminate component structures, interconnected in a statically indeterminate manner. It will subsequently be shown that the specialized terms of the various matrices are for structures that have component structures and interconnections that are statically determinate.

1. Forces on the Component Structure

The above title suggests that the forces on component structures might have different sources. This is indeed true. The correct generation of the corresponding force influence matrices is dependent upon the proper classification of the force sources and their effects. They are so-called "extractor" matrices,[6] because they extract the local force from the generalized loading matrix. These matrices are generated from the equations of static equilibrium, somewhat similarly to the calculation of α_{mh} and β_{mq} for the elements of the component structure. It is the use of these force influence matrices that distinguishes this approach from that of Argyris,[(2)] in which the internal element forces of the connected component structure are used to obtain the work in the components of the complex structure. The use of the new statically obtainable force influence matrices is obviously much simpler to understand as well as to actually implement.

A datum is chosen for each component structure so that the reactions there are statically determinate. At these points, each component structure is joined to another one. All other points on the component represent points where arbitrary loads can be directly introduced. Quantities at these points are designated by the subscript g_i, where g designates the degree of freedom at a point, e.g., a rotation or deflection, and i stands for the interior nature of the point, it being within the boundary of the component or on the boundary, but not connected to other structures there. The forces which correspond to the senses of these degrees of freedom are subscripted with h_i. For example, the numerical designation in an actual analysis of the deformation g_i, which may be a rotation, would be the same as that of the moment h_i if it were applied in the same location.

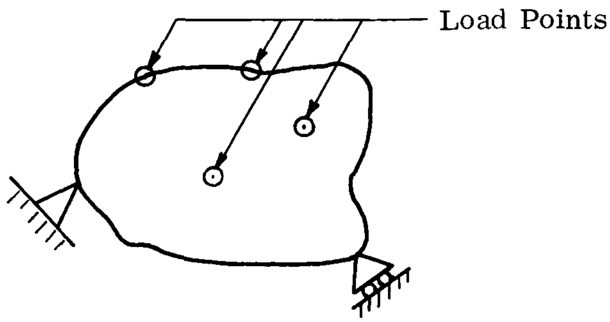


Figure A-7. Typical Load Points g_i, h_i on Component Structure

The points which represent the statically determinate connection of other component structures (i.e., their datums) to the presently considered structure are called g_b. At these points the negative of the reactions of the other components to external loads applied at any point k are introduced (Figure A-8). The subscript b stands for boundary between component structures. At such points, external forces P_{h_b} may be applied with the same sense as the corresponding deformational degrees of freedom there.

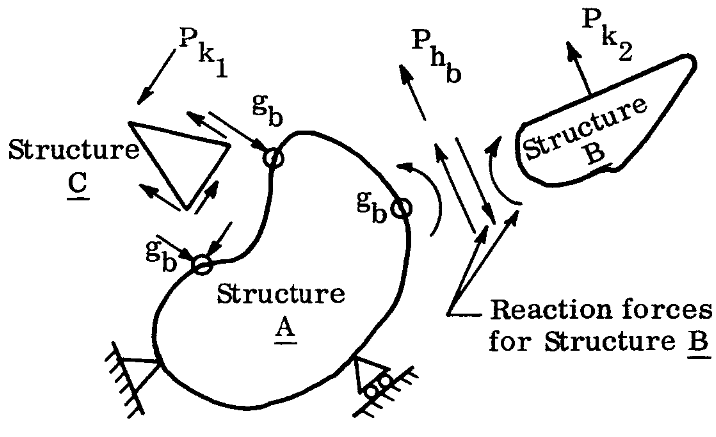


Figure A-8. Typical Load Points g_b

The third type of point is that at which the statically indeterminate coupling forces are introduced. These are called g_t (Figure A-9). These points may be externally loaded, which loads are designated P_{h_t}.

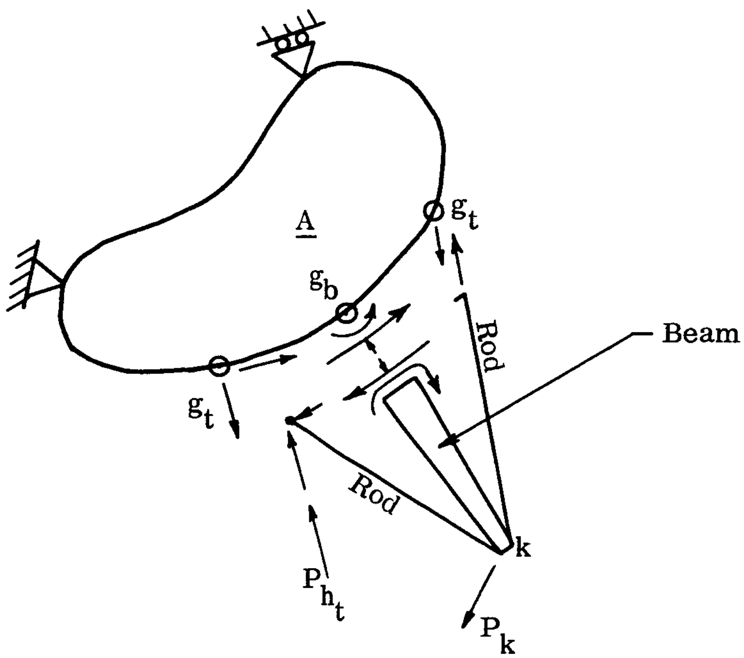


Figure A-9. Redundant Connection of Two Structures

In summary, the point nomenclature is as follows:

$g_{i_{(c)}}$ = Pertains to deformation degree of freedom g at a point within or on boundary of the structure designated (c).

$g_{b_{(c)}}$ = Pertains to deformation degree of freedom at points of statically determinate connection of (c) with other component structures, datum of (c) excluded.

$g_{t_{(c)}}$ = Pertains to deformation degrees of freedom at points of connecting redundant application on structure (c).

Similarily, the corresponding force nomenclature substitutes h for g in the subscripts, other items being held equal so that $h_{i_{(c)}}$, $h_{b_{(c)}}$, $h_{t_{(c)}}$ pertain to forces corresponding in sense and location to their deformation counterpart at $g_{i_{(c)}}$, $g_{b_{(c)}}$ and $g_{t_{(c)}}$, respectively.

It should be noted that the symmetry of the influence coefficient matrix $\left\lceil \gamma_{gh} \right\rfloor$ (Eq. A.28) actually shows that g and h can be used interchangeably.

Having dealt with the force and deformation designations for a component structure, it will now be important to find a nomenclature for the general degrees of freedom and corresponding forces applied anywhere on a complex assembly of component structures. The reason is that we want to describe the response of the complex structure to applied forces in the same way as the response of the component structure was described. This is to say, it is desirable to talk about the degrees of freedom of points that are externally loaded and the degrees of freedom of redundant interconnections separately. Therefore, the following definitions are given

a. Deformation Designations of the Complex Structure

The following subscripts describe the nature of subscripted quantities:

n = Pertains to generalized deformation degrees of freedom anywhere on the complex structure, externally loaded or at which flexibility influence coefficients are required. They include those designated h_i and h_b on the component structures. Excluded are those which correspond to redundants existing at the boundaries between component structures.

s = Pertains to deformation degrees of freedom of the points at the redundant cuts existing between component structures. These degrees of freedom have a subclass, designated by s_p, which are those externally loaded by forces of the same sense and location as the corresponding degree of freedom, or at which flexibility influence coefficients are desired.

b. Loads on the Complex Structure

k, t = Pertains to forces corresponding in sense and location to their deformation counterparts n and s, respectively. This means that the forces designated by k are external forces and those designated by t are the redundants. The forces designated by t_p, corresponding to s_p, are externally applied at the location of interconnecting redundants with the sense of the corresponding degree of freedom. Their positive direction is the same as that for all other externally applied loads which have the same sense.

Using these definitions, a transformation matrix can be constructed which will express the forces applied to a component structure in terms of the forces applied to the complex structure.

The loads P_{h_i} on the structure of Figure A-10 due to forces at the general points are expressed through the transformation matrix $\left[a_{h_i k} \right]$ as follows:

$$\left\{ P_{h_i} \right\} = \left[a_{h_i k} \right] \left\{ P_k \right\}$$

where h_i are points on the component structure and k are all the points on the complex structure where forces are applied, or for which influence coefficients are to be computed.

c. Illustrative Example of $\left[a_{h_i k}\right]$

The loads on the boundary, directly applied, for the structure I of Figure A-10 are described by the matrix $\left[a_{h_i k}\right]$ in terms of the loads $\left\{P_k\right\}$, which is the complete load matrix for the complex structure.

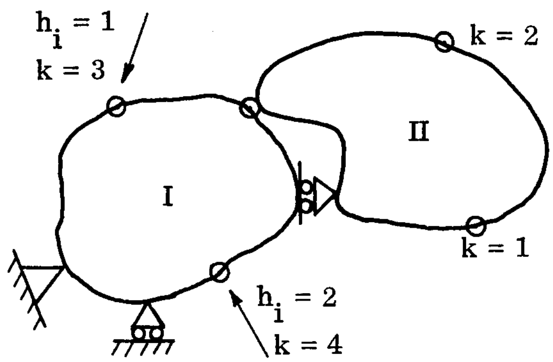


Figure A-10. Structure Loaded by Forces at h_i

$$\left[a_{h_i k}\right]_I = \begin{array}{c|cccc} h_i \backslash k & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \quad ; \quad \left\{P_k\right\} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}$$

so that $\left\{P_{h_i}\right\}_I = \left[a_{h_i k}\right]_I \left\{P_k\right\}$

Thus

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}_I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}$$

For the structure II of Figure A-10:

$$\left[a_{h_i k}\right] = \begin{array}{c|cccc} h_i \backslash k & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \end{array}$$

Thus

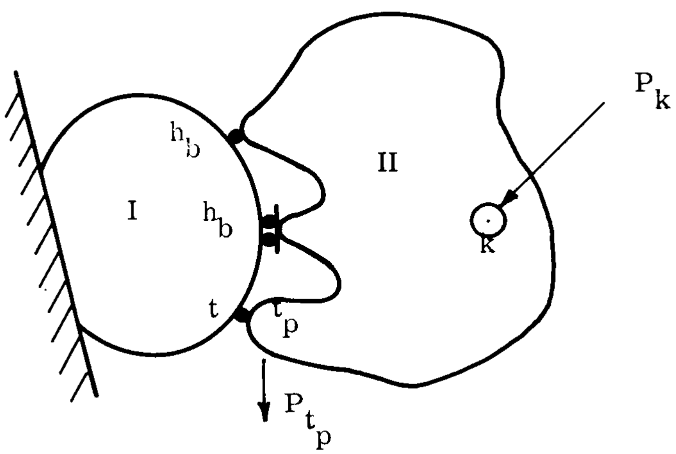
$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}_{II} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}$$

Collectively we can thus write

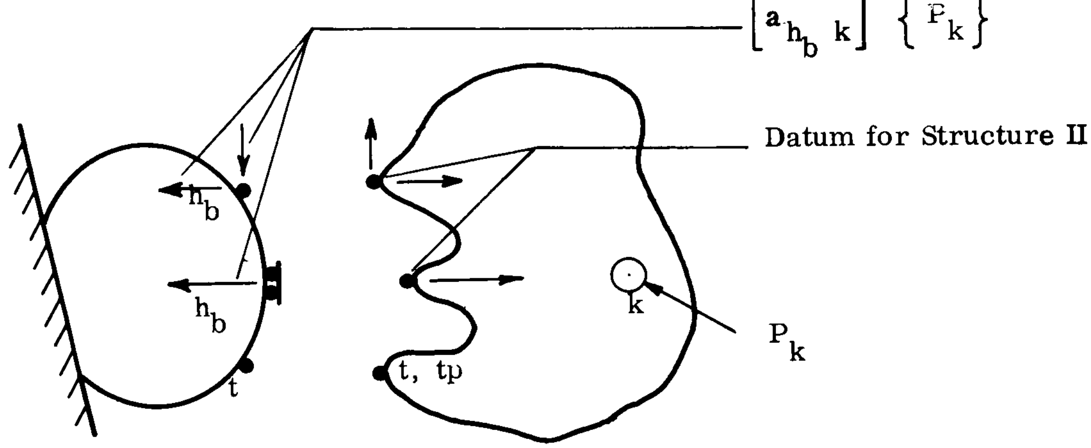
$$\begin{Bmatrix} \{P_{h_i}\}_I \\ \hline \{P_{h_i}\}_{II} \end{Bmatrix} = \begin{bmatrix} \left[a_{h_i k}\right]_I \\ \hline \left[a_{h_i k}\right]_{II} \end{bmatrix} \{P_k\}$$

The forces P_{h_b} have three possible sources as illustrated in Figure A-11. First, there are the reaction forces at h_b due to unit values of forces anywhere at k which form the matrix $\left[a_{h_b k}\right]$. Next, there are reaction forces at h_b due to unit values of forces of type t_p. These are given by $\left[a_{h_b t_p}\right]$. The third contribution to P_{h_b} is the set of reaction forces at h_b due to unit values of redundants at t. The matrix expressing these relations is $\left[b_{h_b t}\right]$. The loads on the structures from these three sources are the sum of their influences:

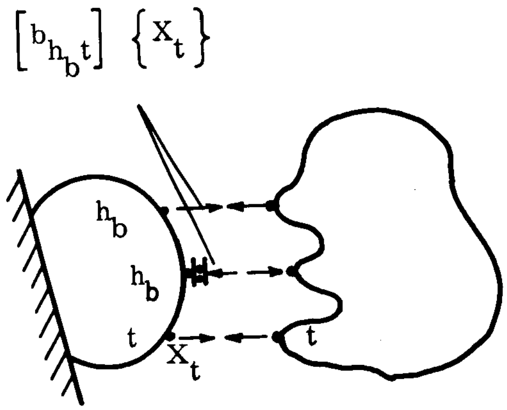
$$\{P_{h_b}\} = \left[\left[a_{h_b k}\right] \Big| \left[a_{h_b t_p}\right] \Big| \left[b_{h_b t}\right] \right] \begin{Bmatrix} P_k \\ \hline P_{t_p} \\ \hline X_t \end{Bmatrix}$$



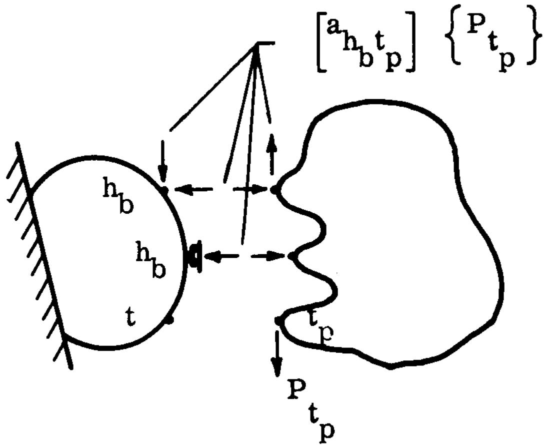
COUPLED STRUCTURES



FORCES AT h_b DUE TO LOADS P_k
ON THE COUPLED STRUCTURE



FORCES AT h_b DUE TO
REDUNDANTS X_t



FORCES AT h_p DUE TO
LOADS P_{t_p}

A-11. Loads on h_b Points of Coupled Component Structures

Finally there are forces at h_t due to forces P_{h_t} applied on <u>one side</u> of the cut face that belongs to the presently considered component structure, and those caused by unit values of the redundants between component structures. The first are given by $\left[a_{h_t t_p}\right]\left\{P_{t_p}\right\}$ and the second by $\left[b_{h_t t}\right]\left\{X_t\right\}$, as shown in Figure A-12.

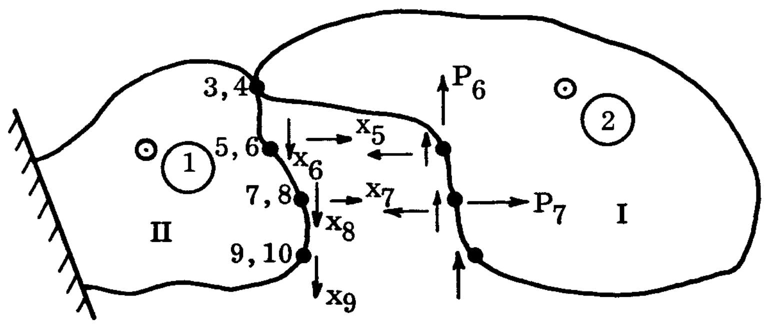


Figure A-12. Forces at h_t Due to Applied Loads and Redundants

$$\left[a_{h_t t_p}\right]_I = \begin{array}{c|c} h_t \backslash t_p & \begin{array}{cc} 6 & 7 \end{array} \\ \hline \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \qquad \left[b_{h_t t}\right]_I = \begin{array}{c|c} h_t \backslash t & \begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \end{array} \\ \hline \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\left[a_{h_t t_p}\right]_{II} = \begin{array}{c|c} h_t \backslash t_p & \begin{array}{cc} 6 & 7 \end{array} \\ \hline \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \qquad \left[b_{h_t t}\right]_{II} = \begin{array}{c|c} h_t \backslash t & \begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \end{array} \\ \hline \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

Thus

$$\left\{P_{h_t}\right\}_{II} = \begin{Bmatrix} P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{Bmatrix} = \left[\begin{array}{c|c} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I \end{bmatrix} \end{array}\right] \left\{\begin{array}{c} P_6 \\ P_7 \\ \hline X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \end{array}\right\}$$

$$\left\{P_{h_t}\right\}_{II} = \begin{Bmatrix} P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{Bmatrix} = \left[\begin{array}{c|c} \begin{bmatrix} 0 \end{bmatrix} & -\begin{bmatrix} I \end{bmatrix} \end{array}\right] \left\{\begin{array}{c} P_6 \\ P_7 \\ \hline X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \end{array}\right\}$$

In summarizing these descriptions, the loads located at various points of the complex structure, cut so that component structures are interconnected in a statically determinate manner, cause forces on any particular component structure (c) which can now be expressed in the matric equation

$$\left\{P_h\right\}_{(c)} = \left\{\begin{array}{c} P_{h_i} \\ \hline P_{h_b} \\ \hline P_{h_t} \end{array}\right\}_{(c)} = \left[\begin{array}{c|c|c} a_{h_i k} & 0 & 0 \\ \hline a_{h_b k} & a_{h_b t_p} & b_{h_b t} \\ \hline 0 & a_{h_t t_p} & b_{h_t t} \end{array}\right]_{(c)} \left\{\begin{array}{c} P_k \\ \hline P_{t_p} \\ \hline X_t \end{array}\right\} \qquad \text{(A.33)}$$

in which the [a] matrices are the forces caused by unit applied loads and [b] are forces due to unit connecting redundants. More briefly stated:

$a_{h_i k}$ = Force on h_i point due to unit load at k.

$a_{h_b k}$ = Force on h_b point due to unit load at k.

$a_{h_b t_p}$ = Force on h_b point due to unit load at t_p.

$a_{h_t t_p}$ = Force at h_t point due to unit load on t_p point.

$b_{h_b t}$ = Force on h_b point due to unit connecting redundant at t.

$b_{h_t t}$ = Force on h_t point due to unit connecting redundant at t.

Let the point designation be generalized so that

i = Location of n and s_p points

j = Location of k and t_p points

that is, i and j designate all points where flexibility influence coefficients are calculated so that, for example,

$$\left\{P_j\right\} = \left\{\begin{array}{c} P_k \\ \hline P_{t_p} \end{array}\right\} \tag{A.34}$$

Then we can let

$$\left[a_{h_j}\right]_{(c)} = \left[\begin{array}{c|c} a_{h_i k} & 0 \\ \hline a_{h_b k} & a_{h_b t_p} \\ \hline 0 & a_{h_t t_p} \end{array}\right]_{(c)} \tag{A.35}$$

and

$$\left[b_{ht} \right]_{(c)} = \left[\begin{array}{c} b_{h_i t} = [0] \\ \hline b_{h_b t} \\ \hline b_{h_t t} \end{array} \right] \qquad \text{(A.36)}$$

Where h now stands for the points h_i, h_b, h_t for which the flexibility matrix γ_{gh} is known from the analysis of each component structures. Then the loads on a particular component structure (c) are:

$$\left\{ \begin{array}{c} P_{h_i} \\ P_{h_b} \\ P_{h_t} \end{array} \right\}_{(c)} = \left\{ P_h \right\}_{(c)} = \left[a_{hj} \,\vdots\, b_{ht} \right]_{(c)} \left\{ \begin{array}{c} P_j \\ \hline X_t \end{array} \right\} \qquad \text{(A.37)}$$

2. The Work of Component Structures

The development of a solution for component structures resulted in a method for calculating the component structure flexibility matrix $\left[\gamma_{gh} \right]_{(c)}$ according to Eq. (A.28). The matrix is initially calculated in the desired arrangement or may be suitably rearranged, so that it is partitioned according to the point categories h_i, h_b, h_t previously defined. Actually this is necessary only for clarity of the derivation. Then the deflections relative to the components datum are:

$$\left\{ \Delta \right\}_{(c)} = \left[\gamma_{gh} \right]_{(c)} \left\{ P_h \right\}_{(c)}$$

or

$$\left\{\begin{array}{c}\Delta_{g_i}\\ \hline \Delta_{g_b}\\ \hline \Delta_{g_t}\end{array}\right\}_{(c)} = \left[\begin{array}{c|c|c}\gamma_{g_i h_i} & \gamma_{g_i h_b} & \gamma_{g_i h_t}\\ \hline \gamma_{g_b h_i} & \gamma_{g_b h_b} & \gamma_{g_b h_t}\\ \hline \gamma_{g_t h_i} & \gamma_{g_t h_b} & \gamma_{g_t h_t}\end{array}\right]_{(c)} \left\{\begin{array}{c}P_{h_i}\\ \hline P_{h_b}\\ \hline P_{h_t}\end{array}\right\}_{(c)} \tag{A.38}$$

The work in the component structure is

$$W_{(c)} = \frac{1}{2}\left[\left\{P_{h_i}\right\}^T\left\{\Delta_{g_i}\right\} + \left\{P_{h_b}\right\}^T\left\{\Delta_{g_b}\right\} + \left\{P_{h_t}\right\}^T\left\{\Delta_{g_t}\right\}\right] \tag{A.39}$$

The work increment due to rigid body motion of the component structure as it displaces due to motion of its datum is zero because the applied forces and the datum reactions are in equilibrium. Therefore, Eq. (A.39) gives one of the components of the total work in the complex structure.

Substituting Eq. (A.37) in Eqs. (A.38) and (A.39), there obtains

$$W_{(c)} = \frac{1}{2}\left[\begin{array}{c|c}P_i & X_s\end{array}\right]\left[\begin{array}{c}a_{ig}\\ \hline b_{sg}\end{array}\right]_{(c)}\left[\gamma_{gh}\right]_{(c)}\left[\begin{array}{c|c}a_{hj} & b_{ht}\end{array}\right]_{(c)}\left\{\begin{array}{c}P_j\\ \hline X_t\end{array}\right\} \tag{A.40}$$

in which

$$\left[P_i \;\vdots\; X_s\right] = \left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\}^T \; ; \; \left[\begin{array}{c} a_{ig} \\ \hline b_{sg} \end{array}\right] = \left[a_{hj} \;\vdots\; b_{ht}\right]^T$$

3. <u>The Work in the Complex Structure</u>

The work in the complex structure is the sum of the works of the components, as given by Eq. (A.40). Thus, for (N) component structures:

$$W_{int} = \sum_{c=1}^{N} W_{(c)}$$

$$W_{int} = \frac{1}{2}\left[P_i \;\vdots\; X_s\right]\left[\begin{array}{cccc} a_{ig_{(1)}} & a_{ig_{(2)}} & \cdots & a_{ig_{(N)}} \\ \hline b_{sg_{(1)}} & b_{sg_{(2)}} & \cdots & b_{sg_{(N)}} \end{array}\right]\left[\begin{array}{cccc} \gamma_{gh_{(1)}} & & & \\ & \gamma_{gh_{(2)}} & & \\ & & \ddots & \\ & & & \gamma_{gh_{(N)}} \end{array}\right]\left[\begin{array}{c|c} a_{hj_{(1)}} & b_{ht_{(1)}} \\ a_{hj_{(2)}} & b_{ht_{(2)}} \\ \vdots & \vdots \\ a_{hj_{(N)}} & b_{ht_{(N)}} \end{array}\right]\left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\} \tag{A.41}$$

The external work of the complex is calculated as was done for the individual component structure. Assume that a flexibility matrix $[\delta]$ of the complex structure exists. The deflections of the external points i are:

$$\left\{\Delta_i\right\} = \left[\delta_{ij} \;\vdots\; \delta_{it}\right]\left\{\begin{array}{c} P_j \\ \hline X_t \end{array}\right\} \tag{A.42}$$

Similarly, the relative deflections of the cut points of statically indeterminate interconnection are

$$\left\{\Delta_s\right\} = \left[\delta_{sj} \mid \delta_{st}\right]\left\{\begin{array}{c}P_j \\ \hline X_t\end{array}\right\} \qquad (A.43)$$

With $\left\{\Delta\right\} = \left\{\begin{array}{c}\Delta_i \\ \hline \Delta_s\end{array}\right\}$, it is obvious that

$$\left\{\Delta\right\} = \left[\begin{array}{c|c}\delta_{ij} & \delta_{it} \\ \hline \delta_{sj} & \delta_{st}\end{array}\right]\left\{\begin{array}{c}P_j \\ \hline X_t\end{array}\right\}$$

Then let the matrix

$$\left[\delta\right] = \left[\begin{array}{c|c}\delta_{ij} & \delta_{it} \\ \hline \delta_{sj} & \delta_{st}\end{array}\right]$$

so that

$$\left\{\Delta\right\} = \left[\delta\right]\left\{\begin{array}{c}P_j \\ \hline X_t\end{array}\right\}$$

The external work is

$$W_{ext} = \frac{1}{2}\left[P_i \mid X_s\right]\left\{\begin{array}{c}\Delta_i \\ \hline \Delta_s\end{array}\right\}$$

so

$$W_{ext} = \frac{1}{2}\left[P_i \mid X_s\right]\left[\begin{array}{c|c}\delta_{ij} & \delta_{it} \\ \hline \delta_{sj} & \delta_{st}\end{array}\right]\left\{\begin{array}{c}P_j \\ \hline X_t\end{array}\right\} \qquad (A.44)$$

Now $\quad W_{int} = W_{ext}$

Let

$$\left[A_{hj}\right] = \begin{bmatrix} a_{hj(1)} \\ a_{hj(2)} \\ \vdots \\ a_{hj(N)} \end{bmatrix} \quad \text{and} \quad \left[B_{ht}\right] = \begin{bmatrix} b_{ht(1)} \\ b_{ht(2)} \\ \vdots \\ b_{ht(N)} \end{bmatrix}$$

Substituting [A] and [B] in Eq. (A.41) and comparing with Eq. (A.44) shows

$$\left[\begin{array}{c} A_{hj}^{T} \\ \hline B_{ht}^{T} \end{array}\right] \left[\diagdown \gamma_{gh} \diagdown\right] \left[\begin{array}{c|c} A_{hj} & B_{ht} \end{array}\right] = \left[\begin{array}{c|c} \delta_{ij} & \delta_{it} \\ \hline \delta_{sj} & \delta_{st} \end{array}\right] \tag{A.45}$$

Compatibility requires that the relative deformations $\{\Delta_s\}$ are equal to zero:

$$\{\Delta_s\} = [\delta_{sj}] \{P_j\} + [\delta_{st}] \{X_t\} = 0 \tag{A.46}$$

Therefore

$$\{X_t\} = -[\delta_{st}]^{-1} [\delta_{sj}] \{P_j\} \tag{A.47}$$

Substituting $\{X_t\}$ in Eq. (A.42) gives

$$\{\Delta_i\} = \left[\begin{array}{c|c} \delta_{ij} & \delta_{it} \end{array}\right] \left\{\begin{array}{c} P_j \\ \hline -\delta_{st}^{-1} \delta_{sj} P_j \end{array}\right\}$$

or

$$\{\Delta_i\} = \left[\delta_{ij} - \delta_{it} \, \delta_{st}^{-1} \, \delta_{sj}\right] \{P_j\} \tag{A.48}$$

Let

$$[\gamma_{ij}] = \left[\delta_{ij} - \delta_{it} \, \delta_{st}^{-1} \, \delta_{sj}\right] \tag{A.49}$$

Then

$$\{\Delta_i\} = [\gamma_{ij}] \{P_j\} \tag{A.50}$$

which shows that $[\gamma_{ij}]$ is, by definition, the flexibility influence coefficient matrix of the complex redundant structure.

Let

$$A_{ig} = A_{hj}^{T}$$

$$B_{sg} = B_{ht}^{T}$$

Then the components of $\left[\gamma_{ij}\right]$ in Eq. (A.49) are obtained from Eq. (A.45) and the substitution of the new terms for A^T, B^T. Thus

$$\delta_{ij} = \left[A_{ig}\right] \left[\diagdown \gamma_{gh} \diagdown\right] \left[A_{hj}\right]$$

$$\delta_{it} = \left[A_{ig}\right] \left[\diagdown \gamma_{gh} \diagdown\right] \left[B_{ht}\right]$$

$$\delta_{sj} = \delta_{it}^{T}$$

$$\delta_{st} = \left[B_{sg}\right] \left[\diagdown \gamma_{gh} \diagdown\right] \left[B_{ht}\right]$$

C. SOLUTION SUMMARY

1. Complex Structure

a. Coupling Redundants

$$\left\{X_t\right\} = -\left[\delta_{st}\right]^{-1}\left[\delta_{sj}\right]\left\{P_j\right\} \qquad \text{(A.47)}$$

b. Forces Applied to Component Structures

Substitution of Eq. (A.47) in (A.37)

$$\left\{P_h\right\} = \left[A_{hj} \;\vdots\; B_{ht}\right]\left[\begin{array}{c} I \\ \hline -\delta_{st}^{-1}\,\delta_{sj} \end{array}\right]\left\{P_j\right\} \qquad \text{(A.51)}$$

c. Displacements

$$\left\{\Delta_i\right\} = \left[\gamma_{ij}\right]\left\{P_j\right\} \qquad \text{(A.50)}$$

2. Component Structure

Substituting $\left\{P_h\right\}$ of Eq. (A.51) in Eqs. (A.30), (A.31), and (A.32), all internal forces in the component structures can be obtained.

Let

$$x_{qh} = -c_{pq}^{-1}\,c_{ph} \qquad \text{(A.52)}$$

(See Equation (A.30))

$$x_{tj} = -\delta_{st}^{-1}\,\delta_{sj} \qquad \text{(A.53)}$$

(See Equation (A.47))

$$f_{hj} = \left[a_{hj} - b_{ht}\,\delta_{st}^{-1}\,\delta_{sj}\right]$$

$$= \left[a_{hj} + b_{ht}\,x_{tj}\right] \qquad \text{(A.54)}$$

(See Equation (A.37), (A.47))

$$f_{mh} = \left[\alpha_{mh} - \beta_{mq}\, c_{pq}^{\;-1}\, c_{ph} \right]$$

or

$$f_{mh} = \left[\alpha_{mh} + \beta_{mq}\, x_{qh} \right] \qquad (A.55)$$

(See Equation (A.31))

Then combining Eqs. (A.52) and (A.54), we obtain component redundants due to unit applied loads as

$$x_{qj} = x_{qh}\, f_{hj} \qquad (A.56)$$

Similarly combining Eqs. (A.54) and (A.55) results in internal forces due to unit applied loads

$$f_{mj} = f_{mh}\, f_{hj} \qquad (A.57)$$

and one obtains the following simplified relationships for the component structure:

Component Structure Redundants

$$\left\{ X_q \right\} = \left[x_{qj} \right] \left\{ P_j \right\} \qquad (A.58)$$

(See Equations (A.30), (A.56))

Component Structure Internal Forces

$$\left\{ F_m \right\} = \left[f_{mj} \right] \left\{ P_j \right\} \qquad (A.59)$$

(See Equations (A.31), (A.57))

Component Structure Deflections

$$\left\{ \Delta_g \right\} = \left[\gamma_{gj} \right] \left\{ P_j \right\} \qquad (A.60)$$

The deflection influence matrix $\left[\gamma_{gj} \right]$ is obtained by choosing those rows of $\left[\gamma_{ij} \right]$ for which $i = g$. The deflections could, of course, have been obtained by choosing the appropriate values from the complex solution, i.e., from $\{\Delta_i\}$, wherever $i = g$. Formally, the matrix $[\gamma_{gj}]$ could be obtained by multiplying $[\gamma_{gh}]$ by the unit load matrix $[f_{hj}]$. Referring to Eq. (A. 54),

$$[\gamma_{gj}] = [\gamma_{gh}] [f_{hj}]$$

D. SPECIAL CASES

1. Statically Determinate Coupling of Statically Indeterminate Components

When no coupling redundants X_t exist, [B] does not exist, as can be seen by referring to Eq. (A.33). Following the development of Eq. (A.45) it becomes obvious that only the matrix $[\delta_{ij}]$ will remain which is given by

$$[\delta_{ij}] = [A_{ig}] [\diagdown \gamma_{gh} \diagdown] [A_{hj}] \tag{A.61}$$

This matrix is also equal to the flexibility influence matrix $[\gamma_{ij}]$ of such a system

$$[\delta_{ij}] = [\gamma_{ij}]$$

The redundants in each component structure are obtained from Eq. (A.58) in which

$$[x_{qj}] = [x_{qh}] [f_{hj}]$$

$$[x_{qh}] = - c_{pq}^{-1} c_{ph}$$

$$[f_{hj}] = [a_{hj}]$$

so that

$$\{X_q\} = - [c_{pq}]^{-1} [c_{ph}] [a_{hj}] \{P_j\} \tag{A.62}$$

The internal forces follow from Eq. (A.59) in which

$$\left[f_{mj}\right] = \left[f_{mh}\right]\left[f_{hj}\right]$$

$$= \left[f_{mh}\right]\left[a_{hj}\right]$$

so that

$$\left\{F_m\right\} = \left[\alpha_{mh} - \beta_{mq}\, c_{pq}^{\;-1}\, c_{ph}\right]\left[a_{hj}\right]\left\{P_j\right\} \qquad \text{(A.63)}$$

2. Statically Determinate Components

If the component structures are statically determinate, their influence coefficient matrices $\left[\gamma_{gh}\right]$ are simply

$$\left[\gamma_{gh}\right]_{S.D.C.} = \left[\alpha_{g\ell}\right]\left[\diagdown \gamma_{\ell m} \diagdown\right]\left[\alpha_{mh}\right] \tag{A.64}$$

because $\left[\beta_{mq}\right]$ does not exist. (S.D.C. means "statically determinate component".) For the same reason, $\left[f_{mj}\right] = \left[\alpha_{mh}\right]\left[f_{hj}\right]$, so that the internal forces are:

$$\left\{F_m\right\}_{S.D.C.} = \left[\alpha_{mh}\right]\left[f_{hj}\right]\left\{P_j\right\} \tag{A.65}$$

In case these component structures are coupled statically determinately, $\left[f_{hj}\right] = \left[a_{hj}\right]$, as before, so that in this case

$$\left\{F_m\right\}_{complex} = \left[\alpha_{mh}\right]\left[a_{hj}\right]\left\{P_j\right\} \tag{A.66}$$

3. Summary

The complete solution of a complex structure consisting of statically indeterminate component structures that are attached to each other in a statically indeterminate manner has been found and is summarized in Eqs. (A.47), (A.50), (A.58) and (A.59). Equations (A.47) and (A.58) allow the calculation of the redundants of the system in two stages, first the coupling redundants and subsequently the component structure redundants. Equation (A.50) gives the displacements of the structure and Eq. (A.59) gives the internal forces in the elements of each component structure. The maximum number of redundants at any stage of the computations can be taylored to suit by providing as many or as few component structures as necessary to limit the size of required inversions, as may be seen from Eqs. (A.47) and (A.58). It can also be seen that the matrix $\lfloor \gamma_{gh} \rfloor$ of each component structure can be obtained by treating it as a complex structure consisting of sub-components, each of which may or may not be redundant, as the configuration dictates.

Finally, several special relationships were given for the cases in which the coupling of the components is statically determinate, or the components are statically determinate, or both. The method presented has, therefore, great usefulness in those applications in which the complex structure is extremely large and can be clerically handled most easily by assigning a component to each of several groups of personnel. Another advantage is that in a large system that may contain several hundred redundants, it will be possible to break the analysis physically into subdivisions so that no inversions of matrices larger than a size for which good precision can be guaranteed will have to be performed.

Even if the structure is statically determinate, the method presented will allow the calculation of the influence coefficient matrix of very large structures in easy stages, as was shown in the final Eqs. (A.64), (A.65) and (A.66) dealing with this degenerate case.

APPENDIX A - PROOF OF EQUATION (A.25)

To prove:

$$\left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] = \left[\begin{array}{c} \alpha_{mh}^{T} \\ \hline \beta_{mq}^{T} \end{array}\right] \left[\diagdown \gamma_{\ell m} \diagdown\right] \left[\begin{array}{c|c} \alpha_{mh} & \beta_{mq} \end{array}\right]$$

From Eq. (A.20) and (A.24),

$$\left[\begin{array}{c|c} P_h^{T} & X_q^{T} \end{array}\right] \left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} = \left[\begin{array}{c|c} P_h^{T} & X_q^{T} \end{array}\right] \left[\begin{array}{c} \alpha_{mh}^{T} \\ \hline \beta_{mq}^{T} \end{array}\right] \left[\diagdown \gamma_{\ell m} \diagdown\right] \left[\begin{array}{c|c} \alpha_{mh} & \beta_{mq} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\}$$

Let

$$\left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\} = \{L\}, \quad \left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] = [c], \quad \left[\begin{array}{c|c} \alpha_{mh} & \beta_{mq} \end{array}\right] = [d],$$

$$\left[\diagdown \gamma_{\ell m} \diagdown\right] = \gamma$$

Then

$$\left[L^{T}\right] \left[c\right] \left[L\right] = \left[L^{T}\right] \left[d^{T} \gamma d\right] \left[L\right]$$

Subtracting equal quantities from both sides:

$$\left[L^{T}\right] \left[c - d^{T} \gamma d\right] \left[L\right] = 0$$

but $\quad L \neq 0$

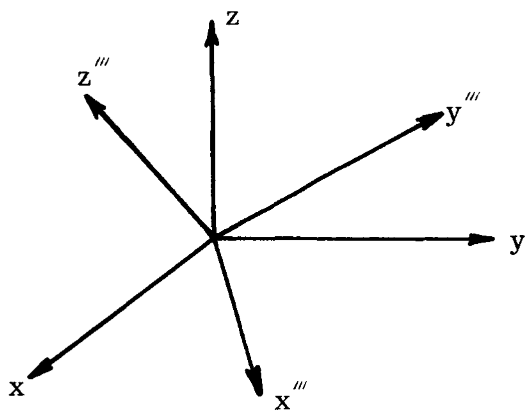
therefore $\quad c = d^{T} \gamma d$

or

$$\left[\begin{array}{c|c} c_{gh} & c_{gq} \\ \hline c_{ph} & c_{pq} \end{array}\right] = \left[\begin{array}{c} \alpha_{mh}^{T} \\ \hline \beta_{mq}^{T} \end{array}\right] \left[\diagdown \gamma_{\ell m} \diagdown\right] \left[\begin{array}{c|c} \alpha_{mh} & \beta_{mq} \end{array}\right]$$

APPENDIX B - AXIS SYSTEM TRANSFORMATIONS OF FORCE AND DEFORMATION VECTORS

Suppose a structural element to exist with orthogonal principal axes x''', y''', z''', oriented arbitrarily with respect to a common Cartesian coordinate system as shown in the following sketch.



Arbitrary Location of Triple Prime Axis System with Respect to Common x, y, z System

Let it be so oriented that the direction cosines of the x''' axis are given by ℓ_x, m_x, and n_x, respectively, being the cosines of the angles between the x'''axis and the x, y,and z axes. Similarly ℓ_y, m_y, and n_y are the direction cosines of the angles between the y''' axis and the x, y, and z axes, respectively. The subscripts z refer to similar quantities pertaining to location of the z axis. Thus it can easily be seen that the transformation of forces from one axis system, x''', y''', z''' to the common Cartesian system is obtained by the matrix multiplication

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{bmatrix} \ell_x & \ell_y & \ell_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \begin{Bmatrix} F_x''' \\ F_y''' \\ F_z''' \end{Bmatrix}$$

Forces and deformations which are alike in sense and direction transform in the same way, through the previously shown direction cosine matrix, in which we let

$$[\,T\,] = \begin{bmatrix} \ell_x & \ell_y & \ell_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix}$$

Methods for obtaining the direction cosines are straightforward and follow from the calculation of unit vectors along chosen body axes. It can be shown that this transformation matrix is orthonormal, so that its inverse is equal to its transpose.

$$[\,T\,]^{-1} = [\,T\,]^{T}$$

In order to ensure the orthogonality of the T vectors, the following orthogonality conditions must be met:

Let $v_i = \begin{Bmatrix} \ell_i \\ m_i \\ n_i \end{Bmatrix}$; $i = x,\ y,\ z$

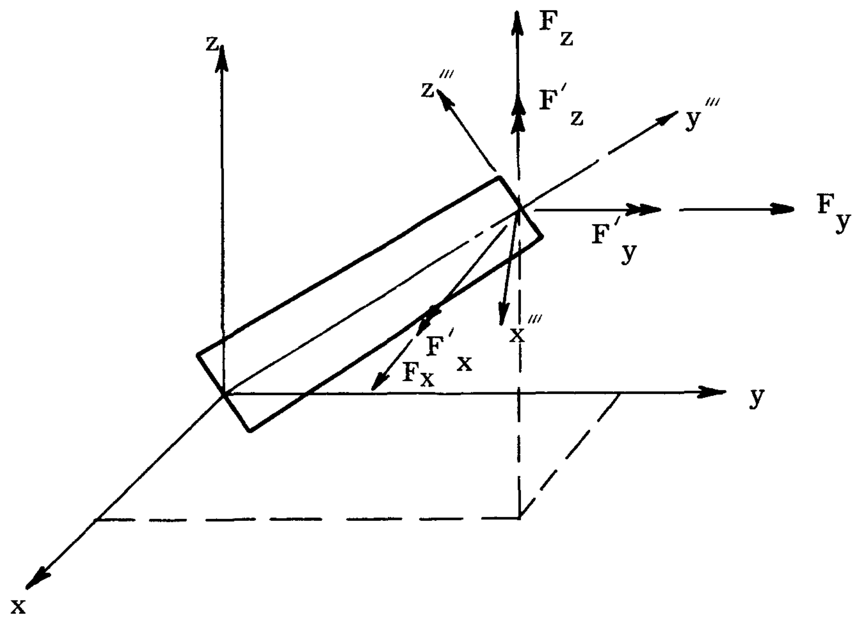
Then $\sum\limits_{n} v_{ni}\, v_{nj} = \delta_{ij}$

where $\delta_{ij} = 1;\ i = j$
$\phantom{\delta_{ij}} = 0;\ i \neq j$

APPENDIX C - EXTERNAL TO LOCAL INTERNAL FORCE TRANSFORMATION

1. Transformation from Common into Local Element Axis System

In the calculation of the influence coefficient matrix of a complex structure it will be necessary to transform forces given in the common x, y, z coordinate system into those acting along the principal axes of the element. Suppose six forces are given at the end of the element shown in the following sketch.



Prismatic Arbitrarily Located Element Subjected to End Forces

Knowing

$$\begin{Bmatrix} F_{x'''} \\ F_{y'''} \\ F_{z'''} \end{Bmatrix} = \left[T\right]^T \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}$$

we arrange the forces which are in the body axis system into two triplets of forces and moments. The resulting transformation yields the formula:

$$\begin{Bmatrix} F'_{x'''} \\ F'_{y'''} \\ F'_{z'''} \\ \hdashline F_{x'''} \\ F_{y'''} \\ F_{z'''} \end{Bmatrix} = \left[\begin{array}{c:c} \begin{bmatrix} T \end{bmatrix}^T & \begin{bmatrix} 0 \end{bmatrix} \\ \hdashline \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} T \end{bmatrix}^T \end{array}\right] \begin{Bmatrix} F'_x \\ F'_y \\ F'_z \\ \hdashline F_x \\ F_y \\ F_z \end{Bmatrix}$$

where $\begin{bmatrix} T \end{bmatrix}^T$ is obtained as previously discussed.

$F'_{x'''}$ = moment about x''' axis

$F'_{y'''}$ = torque about y''' axis

$F'_{z'''}$ = moment about z''' axis

$F_{x'''}$ = transverse shear in x''' axis direction

$F_{y'''}$ = axial force in y''' axis direction

$F_{z'''}$ = shear force in z''' axis direction

2. Transformation of Externally Applied Loads into Internal Element Forces

The forces at the ends of structural elements that are caused by externally applied loads anywhere on the complex structure must be calculated so that the internal work in the elements can be calculated. The use of internal work in the application of the principle of virtual work for the solution of complex redundant structures was shown in Section III. The internal force influence matrices occur in two types. One is the matrix whose elements are the element forces due to unit values of the applied loads; the other type gives the element forces due to unit values of the redundants.

The determination of these matrices obviously requires that the structure be cut until a stable, statically determinate structure remains. Then the internal forces can be obtained from the equations of static equilibrium, which involve only the geometry.

The cutting should be done so that, if possible, the redundants will have the least effect on the internal forces in the structure.

Let $\left[\alpha\right]$ = Influence matrix giving internal forces due to unit values of applied load

$\left[\beta\right]$ = Influence matrix giving internal forces due to unit values of redundant forces acting simultaneously on both sides of one cut

Then the internal forces are:

$$\left\{\begin{array}{c} F_{g'} \\ \hline F_g \end{array}\right\} = \left[\begin{array}{c|c} \alpha_{g'h} & \beta_{g'q} \\ \hline \alpha_{gh} & \beta_{gq} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\}$$

where P_h = external forces at h

X_q = redundant forces at q

But

$$\left\{F_{i'''}\right\} = \left[T\right]^T \left\{F_i\right\}$$

Thus

$$\left\{\begin{array}{c} F'_{g'''} \\ \hline F_{g'''} \end{array}\right\} = \left[\begin{array}{c|c} \left[T\right]^T & \left[0\right] \\ \hline \left[0\right] & \left[T\right]^T \end{array}\right] \left[\begin{array}{c|c} \alpha_{g'h} & \beta_{g'q} \\ \hline \alpha_{gh} & \beta_{gq} \end{array}\right] \left\{\begin{array}{c} P_h \\ \hline X_q \end{array}\right\}$$

These are the desired internal forces in the principal axis system of the structural element. As an example, consider the force transformation for a rod element

$$\left\{ F_{g'''} \right\} = \left[T \right]^T_{rod} \left[\alpha_{gh} \;\vdots\; \beta_{gq} \right] \left\{ \begin{array}{c} P_h \\ \hline X_q \end{array} \right\}$$

where $\left\{ F_{g'''} \right\} = \left\{ F_{y'''} \right\}$

$$\left[T \right]^T_{rod} = \left[\ell \quad m \quad n \right]$$

$$\left\{ \begin{array}{c} F_x \\ F_y \\ F_z \end{array} \right\} = \left[\alpha_{gh} \;\vdots\; \beta_{gq} \right] \left\{ \begin{array}{c} P_h \\ \hline X_q \end{array} \right\}$$

and ℓ, m, n are the direction cosines of the rod axis (y''') with respect to the x, y, z system.

SECTION I - INTRODUCTION

The results obtained using Dr. Lanczos' "Method of Minimized Iterations" on the sample problems so far attempted indicate it is indeed a powerful tool for the solution of the eigenvalue problem. These results show that for the 9 x 9 matrices used for sample problems all nine eigenvalues and eigenvectors were obtained to a minimum of five significant figures. For the case where the matrix had multiple eigenvalues the method determined the number of multiple eigenvalues contained in the matrix.

Each mass station for a three dimensional structure can have up to six degrees of freedom. For large vehicles, such as the Saturn, it may be necessary to use in the neighborhood of one hundred mass stations, including slosh masses, to adequately describe the vehicle. This means the matrix for determining the natural modes of vibration would be of the order of 600.

In view of the large number of mass stations required to adequately describe large space vehicles, it is necessary to apply the present method to a reasonably large problem. This is being done for a sample problem of a Saturn type vehicle.

SECTION II - THE METHOD OF MINIMIZED ITERATIONS

In dynamic analyses, it is required to know the natural frequencies and associated mode shapes of the system. A system may have several modes of vibration with the same frequency or, at the other extreme, the spread of frequencies can be very large. Most practical methods for determining the frequencies and mode shapes of large systems break down under the conditions mentioned above.

The design of space vehicles is an example of a system where both multiple frequencies and large spreads in frequencies can occur. It is necessary to choose a method of analysis which can handle both types of difficulties mentioned. The method chosen is due to Dr. C. Lanczos (Reference 1) and is called, "The Method of Minimized Iterations." This method for the solution of the eigenvalue problem

$$(A - \lambda I)\, x = 0 \tag{B.1}$$

is described below.

If we consider an nth order matrix A, we know there can at most be n linearly independent vectors within the n-dimensional space of the matrix A. Therefore, a linear identity of the following form must exist,

$$b_m + g_1 b_{m-1} + \ldots + g_m b_0 = 0 \tag{B.2}$$

where the b-vectors form a set of base vectors spanning the space of A. When the matrix A has n distinct eigenvalues

$$n = m$$

However, if the matrix A does not have n distinct eigenvalues then

$$1 \leq m \leq n \tag{B.3}$$

This case will be discussed later.

To show the principle of minimized iterations, we first consider the case of symmetric matrices:

$$A = A^* \tag{B.4}$$

where A^* is the transpose of A and we let the multiplication of a vector b by the matrix A equal b'

$$A\,b = b' \tag{B.5}$$

It is now necessary to establish the linear identity (B.2). This identity is approached by choosing a linear combination of the vectors b_0' and b_0 which makes the new vector as small as possible.

The linear combination chosen is

$$b_1 = b_0' - \alpha_0 \, b_0 \tag{B.6}$$

where α_0 is evaluated by the condition

$$(b_0' - \alpha_0 \, b_0)^2 = \text{minimum} \tag{B.7}$$

Then

$$\alpha_0 = \frac{b_0' \, b_0}{b_0^{\,2}}$$

and

$$b_0 \cdot b_1 = b_0 \cdot \left(b_0' - \frac{b_0' \, b_0}{b_0^{\,2}} \, b_0\right) = 0 \qquad \text{(B.8)}$$

i.e., the new vector b_1 is orthogonal to the original vector b_0.

To form b_2 the linear combination chosen is

$$b_2 = b_1' - \alpha_1 \, b_1 - \beta_0 \, b_0 \qquad \text{(B.9)}$$

where again the α_1 and β_0 are obtained from the conditions that b_2^2 be a minimum. The new vector b_2 is orthogonal to both b_0 and b_1.

Forming the b_3 vector we obtain:

$$b_3 = b_2' - \alpha_2 \, b_2 - \beta_1 \, b_1 - \gamma_0 \, b_0 \qquad \text{(B.10)}$$

but because of the orthogonality of b_2 to the previous vectors

$$\gamma_0 = \frac{b_2' \, b_0}{b_0^{\,2}} = \frac{b_2 \, b_0'}{b_0^{\,2}} = 0 \qquad \text{(B.11)}$$

Therefore, the best linear combination has only three terms and every new step of the minimization process has only two correction terms.

The procedure is continued until a set of orthogonal vectors is generated;

$$b_0, b_1, b_2, \ldots, b_{m-1} . \tag{B.12}$$

When the identity (B. 2) is satisfied

$$b_m = 0 . \tag{B.13}$$

If the matrix A is not symmetric; the procedure is modified in the following manner. The operations are performed simultaneously with A and A^*. The operations are the same as previously except the dot products are formed between the b_i and b_i^* vectors. The method can be outlined as follows:

$$b_0 \qquad\qquad b_0^*$$

$$b_1 = b_0' - \alpha b_0 \qquad\qquad b_1^* = b_0^{*\prime} - \alpha_0 b_0^*$$

$$\alpha_0 = \frac{b_0' b_0^*}{b_0 b_0^*} = \frac{b_0^{*\prime} b_0}{b_0^* b_0}$$

$$b_2 = b_1' - \alpha_1 b_1 - \beta_0 b_0 \qquad\qquad b_2^* = b_1^{*1} - \alpha_1 b_1^* - \beta_0 b_0^* \tag{B.14}$$

$$\alpha_1 = \frac{b_1' b_1^*}{b_1 b_1^*} = \frac{b_1^* b_1}{b_1^* b_1}$$

$$\beta_0 = \frac{b_1' b_0^*}{b_0 b_0^*} = \frac{b_1^{*\prime} b_0}{b_0^* b_0} = \frac{b_1^* b_1}{b_0^* b_0}$$

$$b_3 = b_2' - \alpha_2 b_2 - \beta_1 b_1 \qquad\qquad b_3^* = b_2^{*\prime} - \alpha_2 b_2^* - \beta_1 b_1^*$$

etc.

Since, the prime indicates multiplication by the matrix A, the succession of b_i vectors represent a successive set of polynomials. If the letter A is replaced by x we have:

$$
\begin{aligned}
b_0 &= 1 \cdot b_0 \\
b_1 &= (x - \alpha_0)\, b_0 \\
b_2 &= (x - \alpha_1)\, b_1 - \beta_0\, b_0 \\
b_3 &= (x - \alpha_2)\, b_2 - \beta_1\, b_1 \\
&\cdots \\
b_m &= (x - \alpha_{m-1})\, b_{m-1} - \beta_{m-2}\, b_{m-2} = 0
\end{aligned}
\tag{B.15}
$$

The polynomial generated by this procedure is identical to that generated by the procedure known as the "progressive algorithm" of Reference (1). The advantage of the method of minimized iterations lies in the fact that rounding errors do not accumulate. By keeping a constant check on the mutual orthogonality of the b_i and b_i^* vectors, any lack of orthogonality caused by rounding errors are immediately corrected by means of the correction term

$$\epsilon_{ij} = \frac{b_i \cdot b_j}{b_j^2}\, b_j \tag{B.16}$$

The biorthogonality of the vectors b_i and b_i^* leads to an explicit solution of the eigenvalue problem in terms of the b_i vectors. The method by which the b_i vectors were generated gives

$$b_i = p_i(\mu_1)\, u_1 + p_i(\mu_2)\, u_2 + \dots + p_i(\mu_m)\, u_m \tag{B.17}$$

Dotting b_i with u_k^* we obtain

$$b_i \cdot u_k^* = p_i(\mu_k)\, u_k \cdot u_k^* \tag{B.18}$$

If the u_i are expanded in terms of the b_i then

$$u_i = \alpha_{i0}\, b_0 + \alpha_{i1}\, b_1 + \dots + \alpha_{im-1}\, b_{m-1} \tag{B.19}$$

Dotting the u_i with b_k^* gives

$$\alpha_{ik} = \frac{u_i \cdot b_k^*}{b_k \cdot b_k^*} \tag{B.20}$$

The "norm" of b_k is given by σ_k :

$$\sigma_k = b_k \cdot b_k^* \tag{B.21}$$

while the norm of u_k is left arbitrary.

The expansion (B.19) then becomes:

$$u_i = \frac{b_0}{\sigma_0} + p_1(\mu_i)\,\frac{b_1}{\sigma_1} + p_2(\mu_i)\,\frac{b_2}{\sigma_2} + \dots + p_{m-1}(\mu_i)\,\frac{b_{m-1}}{\sigma_{m-1}} \tag{B.22}$$

where the u_i are the eigenvectors and the b_i are the vectors generated in the method of minimized iterations.

The u_i^* vectors are generated in the same manner

$$u_i^* = \frac{b_0^*}{\sigma_0} + p_1(\mu_i)\frac{b_1^*}{\sigma_1} + \dots + p_{m-1}(u_i)\frac{b_{m-1}^*}{\sigma_{m-1}} \tag{B.23}$$

When the matrix A contains multiple eigenvalues the complete characteristic equations can not be fully established since the multiple roots behave as single roots. In this case, the characteristic equation is of lower order than that of the matrix A.

In order to obtain multiple roots, the following procedure is necessary. A starting vector b_{01} is chosen and the procedure is carried to completion. A second starting vector b_{02} is then chosen and again the procedure is carried to completion. An examination is then made of the eigenvectors, those eigenvectors that are identical for every starting vector are associated with the single eigenvalues. If those eigenvectors associated with a given eigenvalue are different then that eigenvalue is multiple. To determine the degree of multiplicity it is necessary to take sufficient starting vectors until the identity

$$x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n = 0 \tag{B.24}$$

is satisfied, where the x_i are the eigenvectors associated with a given eigenvalue for each starting vector.

SECTION III - TEST PROBLEMS USING THE METHOD OF MINIMIZED ITERATIONS

To test the accuracy of the method described in the previous section a simple beam was chosen as a test problem. The mode shapes and frequencies for a simple beam can be calculated from the following formulae (reference (2)).

$$y = \sin \frac{n \pi x}{\ell} \tag{B.25}$$

$$w_n = \frac{n^2 \pi^2}{\ell^2} \sqrt{\frac{EI}{\rho}} \tag{B.26}$$

It was decided to use a nine degree of freedom system, therefore, the first nine eigenvectors and eigenvalues were calculated using Eqs. (B.25) and (B.26). Then using the relationship (Reference (3))

$$A = [\, y \,] [\, \lambda \,] [\, y \,]^{-1} \tag{B.27}$$

we can obtain a matrix A, the eigenvectors and eigenvalues of which are known. The eigenvectors and eigenvalues calculated from Eqs. (B.25) and (B.26) are shown in tables 1 and 2. Using the method of minimized iterations the matrix A was operated upon to obtain its eigenvectors and eigenvalues. The results obtained by this method are shown in tables 3 and 4.

A comparison of the exact eigenvalues with those obtained from the method of minimized iterations show that the latter agree with the exact results to five significant places. These results are excellent, especially since the spread between the largest and smallest eigenvalue is approximately 7000:1. The eigenvectors associated with these eigenvalues are accurate to at least five significant figures and in most cases more.

A test sample problem was devised to test the ability of the method to determine multiple eigenvalues. In this case, the same eigenvectors were chosen as in the first problem but an arbitrary set of eigenvalues were chosen. The eigenvalues chosen for this problem are shown in table 2. Again Eq. (B.27) was used to obtain the A matrix and the method of minimized iterations used to obtain the eigenvalues and eigenvectors of this A matrix. However, for this problem three different starting vectors were used to determine which were the multiple eigenvalues. The results obtained for the second problem are shown in tables 5 through 8.

It should be noted the second starting vector was inadvertently chosen to be orthogonal to the fifth eigenvector of the A matrix. The result of this was to make the matrix A seem to be of one order less. For this reason, any problem should always be run with two distinct trial vectors.

An examination of the results show that for those eigenvalues which are distinct the eigenvectors associated with them are identical for every trial vector. For the multiples eigenvalues the eigenvectors appear as linear combinations of the original eigenvectors. It is necessary to apply Eq. (B.24) to determine the number of each multiple eigenvalue.

A comparison of the computed eigenvalues with the exact values show the two agree within the number of places used. The eigenvectors associated with the distinct eigenvalues also show agreement to approximately seven places.

TABLE 1

EIGENVECTORS FOR TEST PROBLEM

CALCULATED FROM EQUATION (B. 25)

.13819660	.26286556	.36180340	.42532541	.44721361	.42532541	.36180340	.26286556	.13819660
.26286556	.42532541	.42532541	.26286556	0	-.26286556	-.42532541	-.42532541	-.26286556
.36180340	.42532541	.13819660	-.26286556	-.44721361	-.26286556	.13819660	.42532541	.36180340
.42532541	.26286556	-.26286556	-.42532541	0	.42532541	.26286556	-.26286556	-.42532541
.44721361	0	-.44721361	0	.44721361	0	-.44721361	0	.44721361
.42532541	-.26286556	-.26286556	.42532541	0	-.42532541	.26286556	.26286556	-.42532541
.36180340	-.42532541	.13819660	.26286556	-.44721361	.26286556	.13819660	-.42532541	.36180340
.26286556	-.42532541	.42532541	-.26286556	0	.26286556	-.42532541	.42532541	-.26286556
.13819660	-.26286556	.36180340	-.42532541	.44721361	-.42532541	.36180340	-.26286556	.13819660

TABLE 2

EXACT EIGENVALUES USED IN TEST PROBLEMS

Simple Beam	Multiple Eigenvalues
410.6388×10^{-4}	7.0000000
25.66495×10^{-4}	7.0000000
5.0696208×10^{-4}	6.0000000
1.6040597×10^{-4}	5.0000000
$.65702292 \times 10^{-4}$	4.0000000
$.3168513 \times 10^{-4}$	3.0000000
$.17102844 \times 10^{-4}$	2.0000000
$.10025373 \times 10^{-4}$	1.0000000
$.06258791 \times 10^{-4}$	1.0000000

TABLE 3

EIGENVECTORS OBTAINED FROM

METHOD OF MINIMIZED ITERATIONS

SIMPLE BEAM

.1381966	.2628656	.3618034	.4253255	.4472134	.4253227	.3618056	.2628703	.1382226
.2628656	.4253254	.4253253	.2628657	.0000003	-.2628615	-.4253178	-.4253300	-.2628944
.3618034	.4253254	.1381967	-.2628661	-.4472137	-.2628678	.1381778	.4253273	.3618179
.4253254	.2628656	-.2628656	-.4253249	0	.4253233	.2628827	-.2628657	-.4253248
.4472137	0	-.4472135	0	.4472133	.0000052	-.4472182	-.1438830	.4472031
.4253254	-.2628656	-.2628656	.4253249	.0000004	-.4253291	.2628595	.2628641	-.4253113
.3618034	-.4253254	.1381967	.2628661	-.4472138	.2628651	.1382033	-.4253216	.3617924
.2628656	-.4253254	.4253253	-.2628657	0	.2628677	-.4253259	.4253215	-.2628605
.1381966	-.2628656	.3618034	-.4253255	.4472138	-.4253265	.3618004	-.2628638	.1381953

TABLE 4

EIGENVALUES OBTAINED FROM METHOD

OF MINIMIZED ITERATIONS

Simple Beam

410.6388×10^{-4}

25.66495×10^{-4}

5.069621×10^{-4}

1.604057×10^{-4}

$.6570229 \times 10^{-4}$

$.3168533 \times 10^{-4}$

$.1710268 \times 10^{-4}$

$.1002538 \times 10^{-4}$

$.06258738 \times 10^{-4}$

TABLE 5

EIGENVECTORS OBTAINED FROM

METHOD OF MINIMIZED ITERATIONS

MULTIPLE EIGENVALUES

First Trial Vector

.2969791	.3618034	.4253254	.4472136	.4253254	.3618034	.2969791
.4987910	.4253254	.2628656	0	-.2628656	-.4253254	-.4987910
.5448309	.1381966	-.2628655	-.4472136	-.2628655	.1381966	.5448309
.4305920	-.2628656	-.4253254	0	.4253254	.2628656	-.4305920
.2081069	-.4472136	0	.4472135	0	-.4472136	.2081069
-.0347492	-.2628656	.4253254	0	-.4253254	.2628656	.0347492
-.2081068	.1381965	.2628656	-.4472137	.2628656	.1381966	-.2081069
-.2541467	.4253254	-.2628655	0	.2628656	-.4253254	.2541467
-.1683620	.3618034	-.4253254	.4472136	-.4253254	.3618034	-.1683620

TABLE 6

EIGENVECTORS OBTAINED FROM

METHOD OF MINIMIZED ITERATIONS

MULTIPLE EIGENVALUES

Second Trial Vector

.2962610	.3618034	.4253254	.4253254	.3618034	.2962610
.5000000	.4253254	.2628656	-.2628656	-.4253254	-.5000000
.5520147	.1381966	-.2628655	-.2628655	.1381966	.5520147
.4472136	-.2628656	-.4253254	.4253254	.2628656	-.4472136
.2351141	-.4472136	0	0	-.4472136	.2351141
0	-.2628656	.4253254	-.4253254	.2628656	0
-.1715921	.1381965	.2628656	.2628656	.1381966	-.1715921
-.2236068	.4253254	-.2628655	.2628656	-.4253254	.2236068
-.1509525	.3618034	-.4253254	-.4253254	.3618034	-.1509525

TABLE 7

EIGENVECTORS OBTAINED FROM

METHOD OF MINIMIZED ITERATIONS

MULTIPLE EIGENVALUES

Third Trial Vector

.2897658	.3618034	.4253255	.4472136	.4253254	.3618034	.2897658
.4942882	.4253254	.2628656	0	-.2628656	-.4253254	-.4942882
.5583936	.1381966	-.2628655	-.4472136	-.2628655	.1381966	.5583936
.4758070	-.2628657	-.4253254	0	.4253254	.2628656	-.4758070
.2897659	-.4472139	-.0000002	.4472137	-.0000002	-.4472137	.2897659
.0753604	-.2628659	.4253252	0	-.4253254	.2628656	-.0753604
-.0895425	.1381962	.2628652	-.4472134	.2628653	.1381963	-.0895426
-.1536480	.4253251	-.2628657	0	.2628656	-.4253253	.1536480
-.1106807	.3618032	-.4253256	.4472137	-.4253255	.3618033	.1106807

TABLE 8

EIGENVALUES OBTAINED FROM METHOD

OF MINIMIZED ITERATIONS

Multiple Eigenvalues

1st Trial Vector	2nd Trial Vector	3rd Trial Vector
7.000000	7.000000	7.000000
6.000000	6.000000	6.000000
5.000000	5.000000	5.000000
4.000000	3.000000	4.000000
3.000000	2.000000	3.000000
2.000000	1.000000	2.000000
1.000000		1.000000

PART C

SAMPLE PROBLEM

SECTION I - INTRODUCTION

The method of minimized iterations has been checked using the test problems shown in Part B but these are for small scale systems. To further check the method, the tank configuration shown in Figure C-1 has been analyzed.

In Figure C-2 is shown the idealized spring mass system which was analyzed. Each mass point was considered to have three translational degrees of freedom. Influence coefficients required in the calculation of the dynamic matrix were obtained using the method described in Part A. With these influence coefficients and a corresponding mass matrix, a three dimensional "free-free" dynamic matrix was calculated. The modes and frequencies of this matrix were then obtained using the method of minimized iterations.

As shown in Figure C-1, the analytic model consists of a lower stage with a central tank, four peripheral tanks and an upper stage with a single tank. Each tank is described by four mass points. (See Figure C-2.) The lower stage tanks are connected by a beam network at the upper and lower ends of the tanks. For this analysis the connecting beams are considered massless but their elastic properties are accounted for in the influence coefficients.

SECTION II - CALCULATION OF FLEXIBILITY MATRIX

A. DESCRIPTION OF STRUCTURE

Influence coefficients are calculated for a multiple tank booster system, Figure C-1. They are calculated for the three deflection degrees of freedom at each mass point, which are defined at four points on each tank centerline as shown in Figure C-2.

The structure consists of five tanks in the main stage and one in the upper stage. The main stage tanks rest on a network of beams called the tail spider, and a similar spider is employed between the upper and main stages. The spider layout is shown in Figure C-3.

Three of the lower five tanks contain lox. They are connected to each other through the beams of the spiders and therefore differential longitudinal expansion is accordingly somewhat restricted between them. The two tanks containing fuel, located diametrically opposite from each other, can extend freely in the axial direction because they are equipped with a sliding connection at the interstage spider. The tanks are long and slender and are therefore considered to act as beams.

B IDEALIZATION OF STRUCTURE

1. General

The structure idealization is shown in Figure C-2. The datum of the total structure is formed by the four crossing points of the interstage spider. The beam sections between these points, which are fixed in space, may bend but not extend. The four vertical restraints are shown in Figure C-4. The upper stage tank VIII and the main stage center tank I are cantilevered from the plane formed by these points.

The tail spider is attached to the bottom end of tank I at the four crossing points, which are assumed to remain in a plane. The beam sections of the spider between these points may bend but not extend. The outer tanks are suspended between the ends of the beams. There are both determinate and redundant connecting forces between the outer tanks and the spiders. These are shown in Figure C-5 for lox tank IV, which is typical.

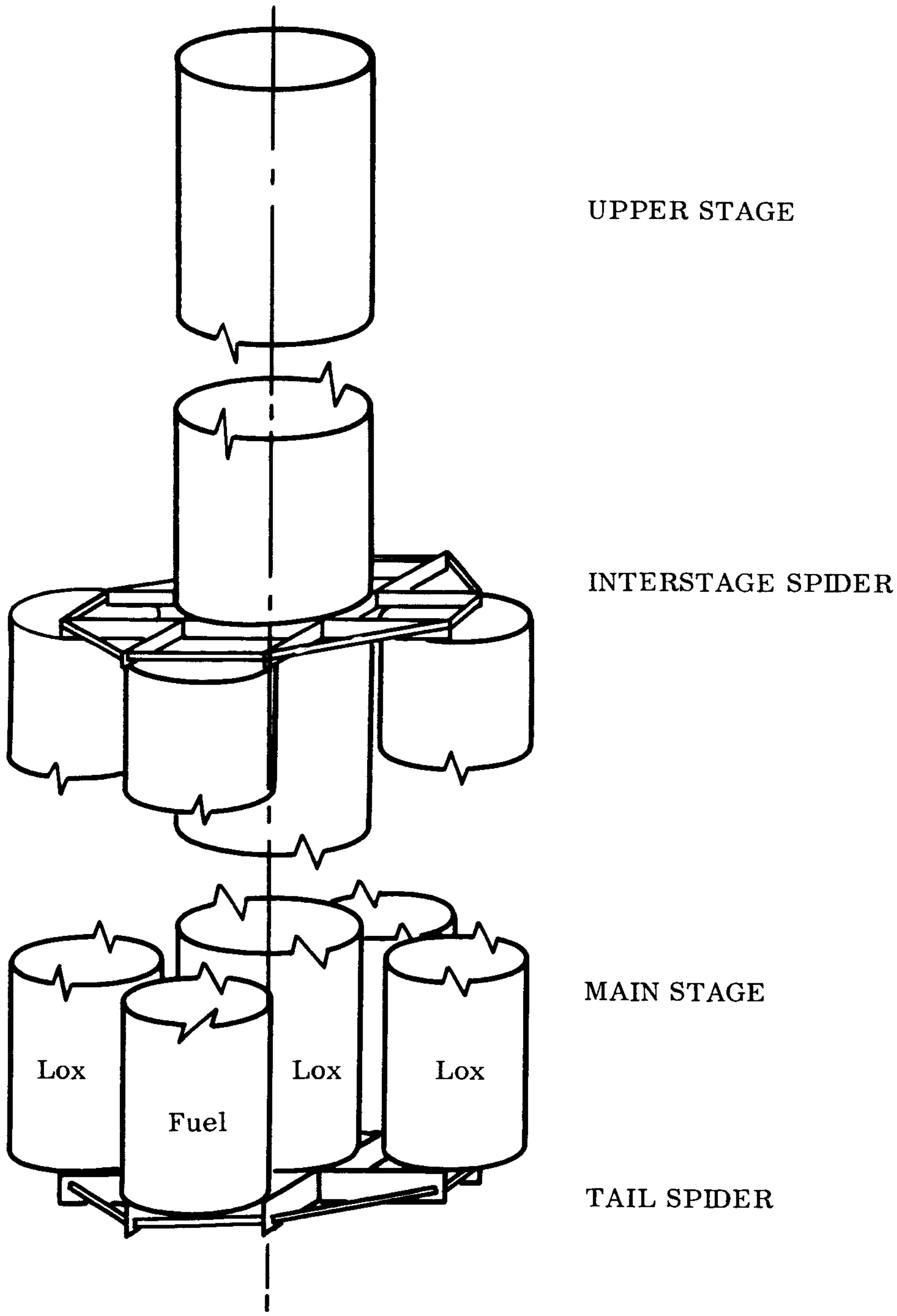
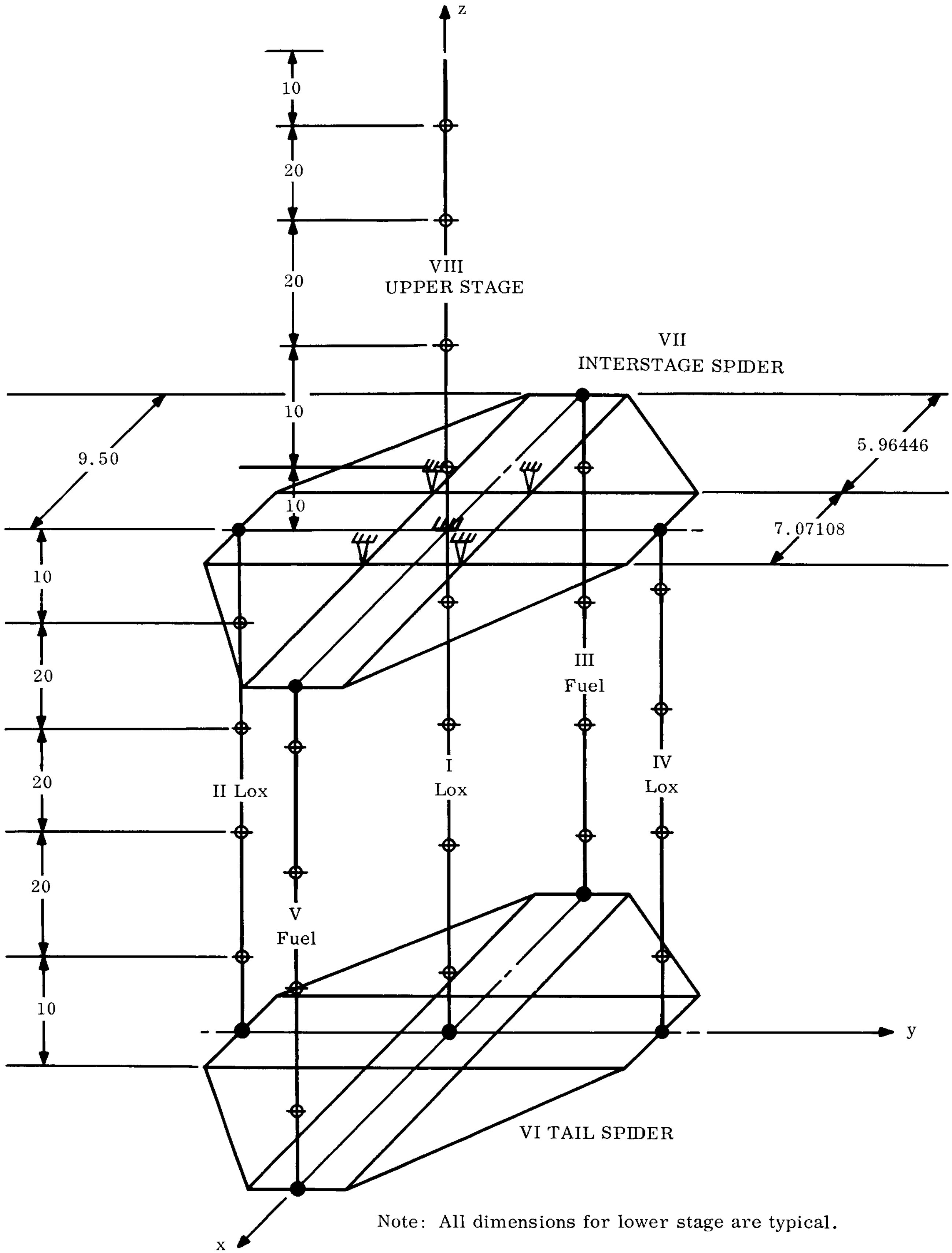


Figure C-1. Dynamic Model for Multiple Tank Booster



Note: All dimensions for lower stage are typical.

Figure C-2. Representation of Sample Problem
a. Idealization of Tank Structure

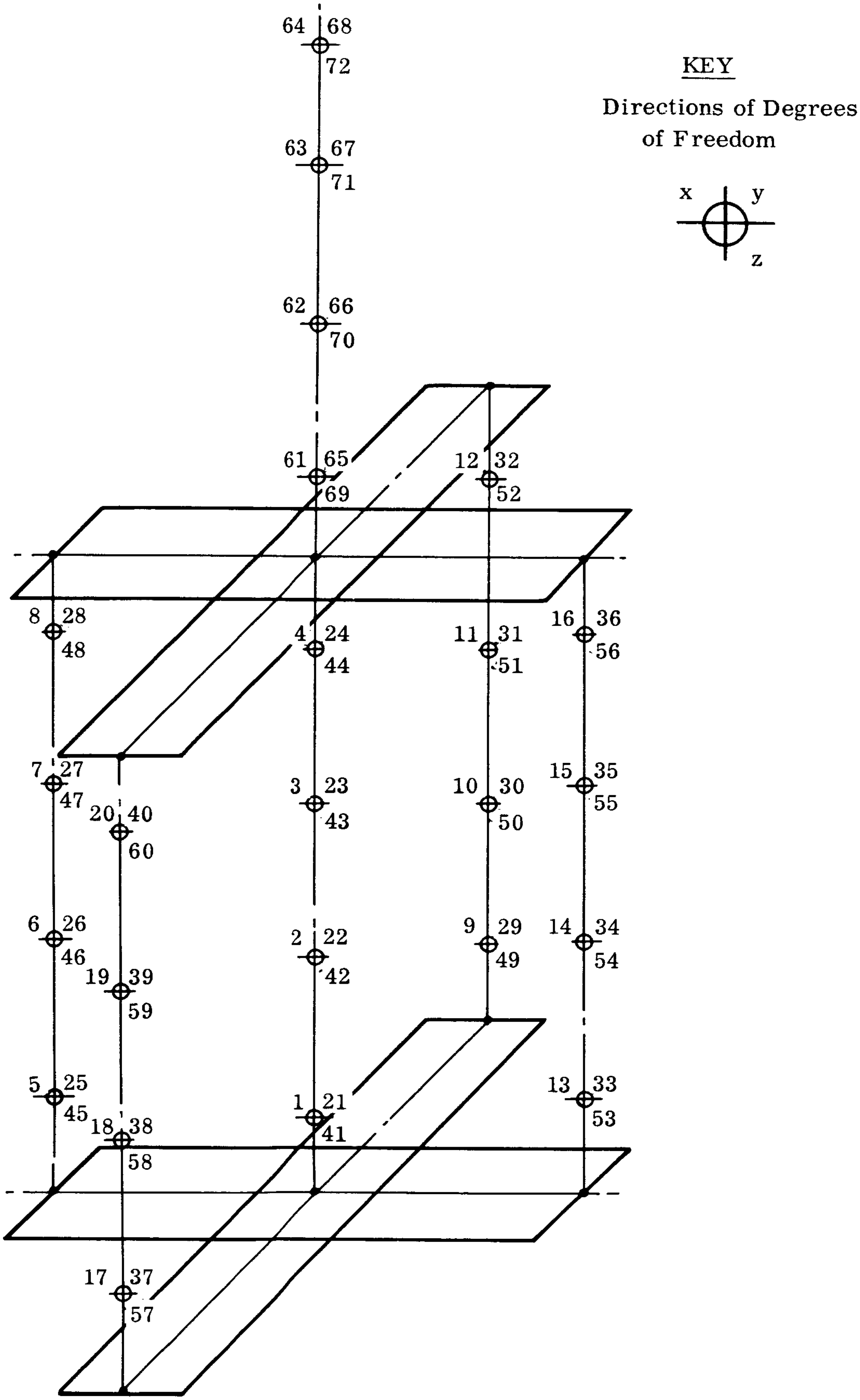


Figure C-2. Representation of Sample Problem
b. Degree of Freedom Identification of Mass Points

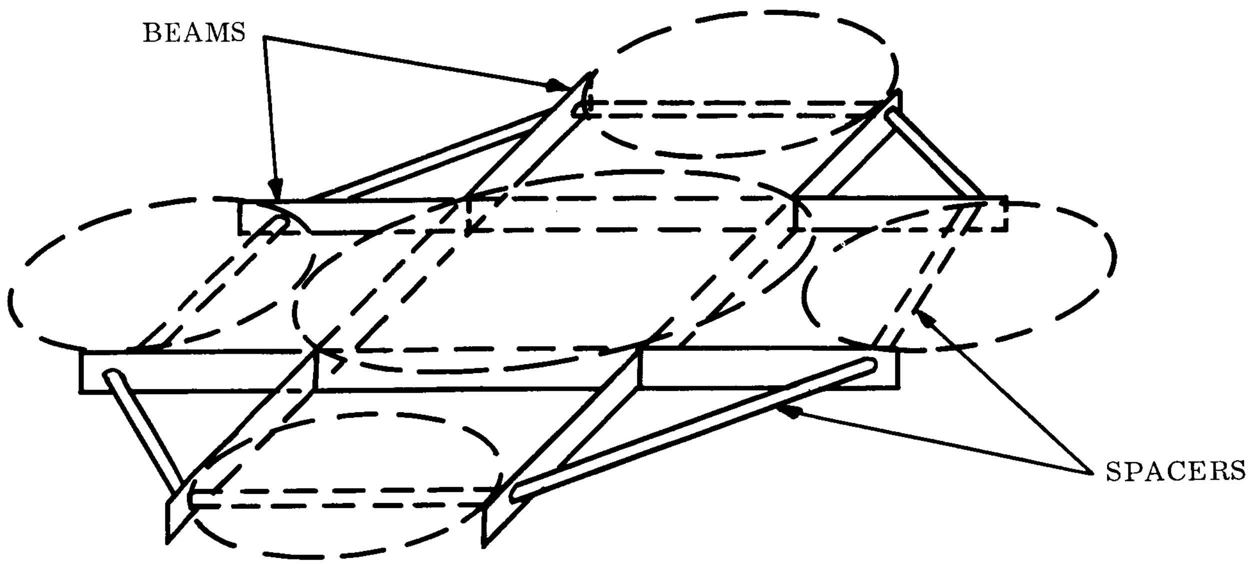


Figure C-3. Spider Beams and Spacers

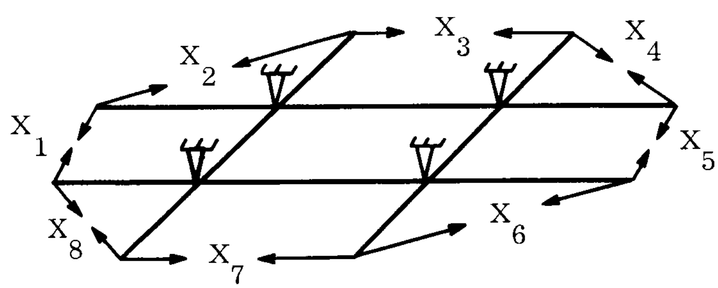


Figure C-4. Internal Redundants X_q of Spiders

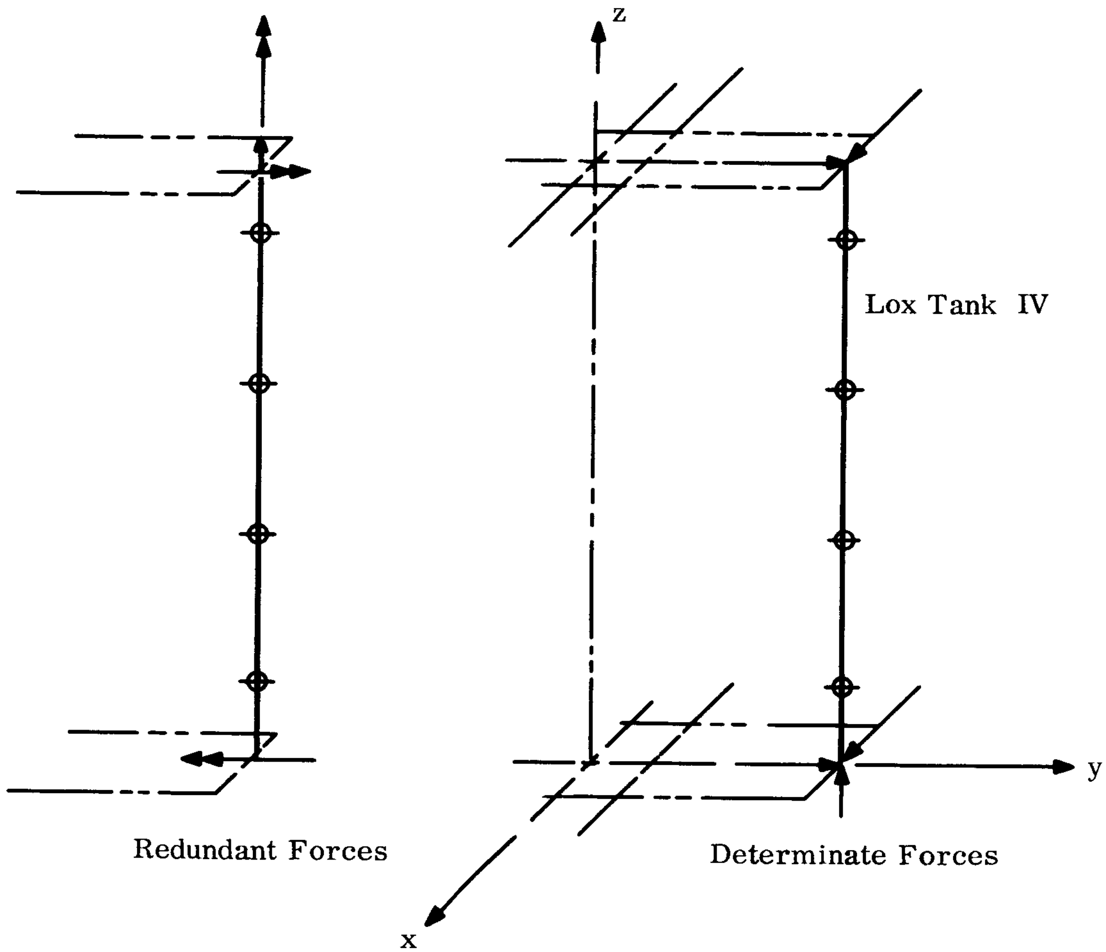


Figure C-5. Connecting Forces of Tank IV

Section properties of the tanks, which are assumed to act as beams, and the beams and spacers comprising the spiders, are given in Table 9. The total mass of each tank is divided into four equal parts located on the centerlines.

TABLE 9

SECTION PROPERTIES

Tanks

No.	L(in)	D(in)	t(in)	EI(lb-in^2)	GJ(lb-in^2)	AE(lb)
I	80	10	.025	9.82×10^7	7.86×10^7	7.86×10^6
II	80	7.07	.032	4.45×10^7	3.56×10^7	7.09×10^6
III	80	7.07	.020	2.78×10^7	2.24×10^7	4.44×10^6
IV	80	7.07	.032	4.45×10^7	3.56×10^7	7.09×10^6
V	80	7.07	.020	2.78×10^7	2.24×10^7	4.44×10^6
VIII	80	10	.025	9.82×10^7	7.86×10^7	7.86×10^6

Interstage Spider

	EI(lb-in^2)	EI, $_{Transv.}$ (lb-in^2)	AE(lb)
Beams (all)	10^6	10^4	$.16 \times 10^7$
Spacers (all)	-	-	$.2 \times 10^5$

Tail Spider

	EI(lb-in^2)	EI, $_{Transv.}$ (lb-in^2)	AE(lb)
Beams (all)	2.25×10^7	10^5	$.75 \times 10^7$
Spacers (all)	-	-	10^6

2. Redundants Between Components

Figure C-6 shows the idealization of the outer tank connections to the interstage spider. The connections at the tail spider for all four tanks are identical to the connections of the lox tanks to the interstage spider. Only the upper ends of the fuel tanks have a different type connection which allows free axial expansion and small rotations around the x-axis. All other connections resist these rotations and develop moments, which are selected as redundants. As the free expansion of the lox tanks is impeded by the pinned joints at both ends, the axial force in each outer tank is also selected as a redundant. The torsional moments in each of the outer tanks form the last set of redundant forces between the components. All of these are shown in Figure C-7.

3. Internal Redundants of Component Structures

The spiders are the only component structures which are statically indeterminate. The axial forces in the peripheral spacers are selected as the internal redundants X_q in each spider.

C. EXPLANATION OF MATRICES

1. $[\alpha]$ Matrix

Each of the components is separated into elements such as cantilevers, simply supported beams, or rods. The elements' ends are at the points of connection to other elements or at load points.

The m-designations are provided for the selected internal forces of the components. These forces may be thought of as external loads acting on the elements into which the components are separated. Component I, the middle tank of the main stage, is separated into five elements, each cantilevered at the upper end as shown in Figure C-8. Each of these elements can support two shears, an axial load, two bending moments, and a torque. Element A is the one at the lower end and its free end forms the connection to the other structure. It is possible to have all six forces at this connection and the element can resist all. Therefore, the first six m-designations correspond to these forces. At the lowest mass point on the tank there may likewise be all six types of internal forces, so that these are basically the next m-designations at the end of element B, etc.

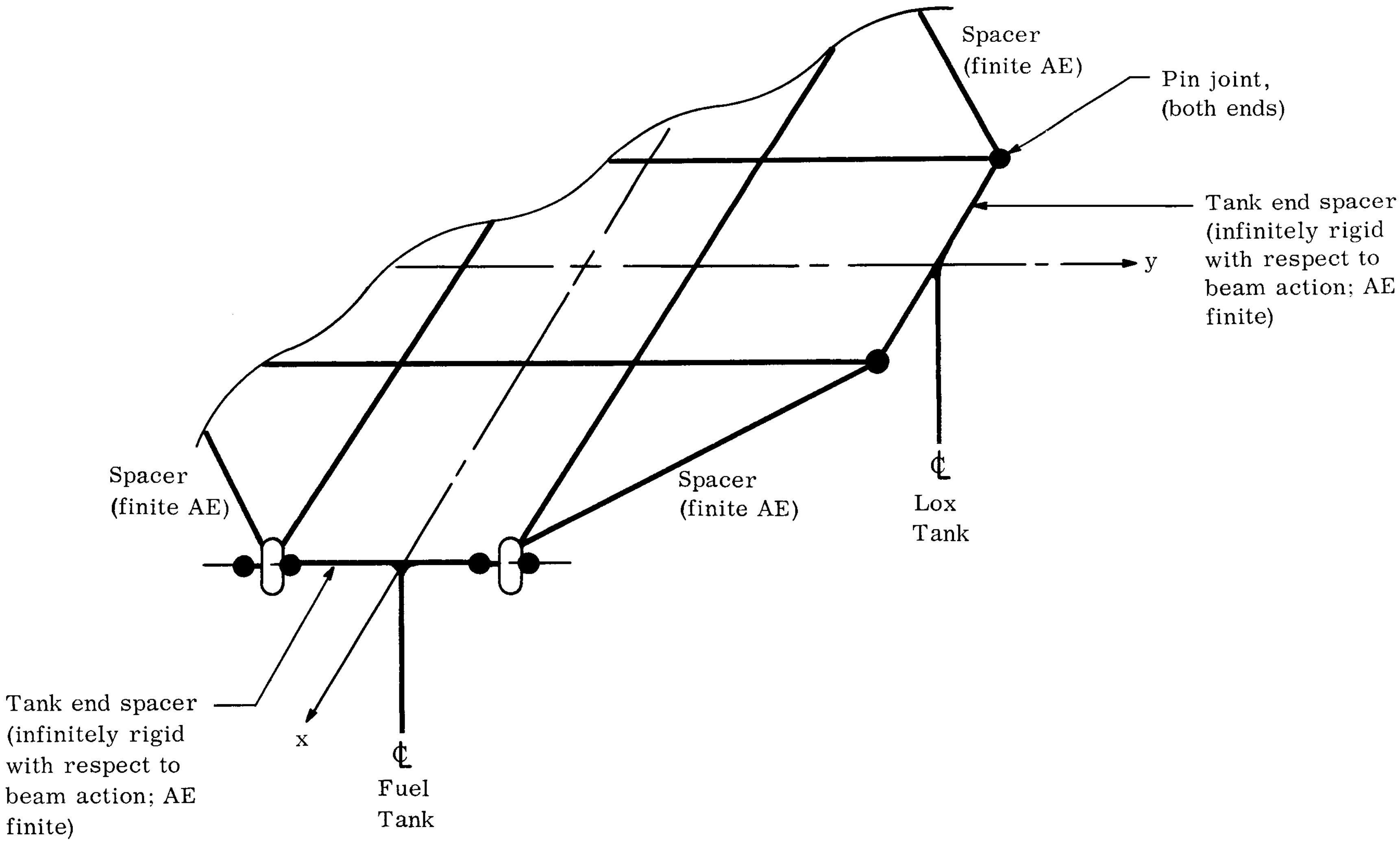


Figure C-6. Outer Tank Connections to Interstage Spider

However, as it is not necessary in this problem to know the twists of each individual element, but rather only the total twist of the component, element A has assigned to it the total torsional flexibility of the component. The remaining elements have no twisting degree of freedom. The procedure for the other tanks is similar.

The spider beams are separated into cantilever elements and simply supported beam elements. The possible internal forces on the extending cantilevers are two shears and an axial load. The simply supported beam elements are located between the ends of the cantilevers. On these elements, there occur bending moments about both axes at each end. These are shown in Figure C-9.

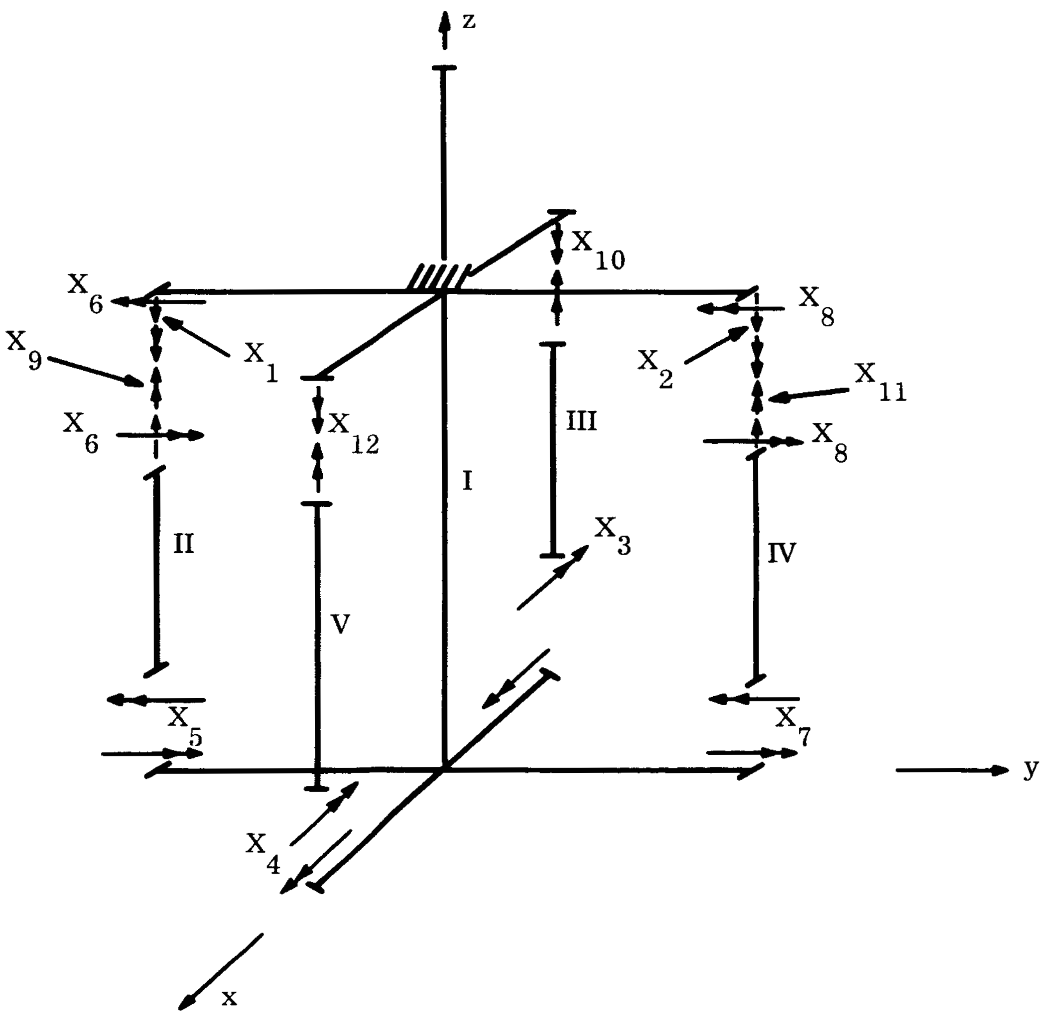
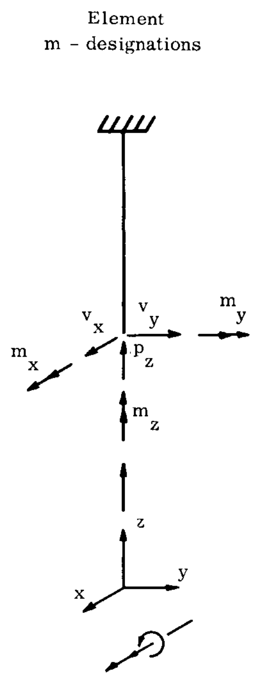


Figure C-7. Redundant Forces X_t in the Idealized Structure

Element	Force type	m \ h	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
			h_b						h_i											
A	v_x	1	●																	
	v_y	2		●																
	p_z	3			●															
	m_x	4				●														
	m_y	5					●													
	m_z	6						●												
B	v_x	7	●						●											
	v_y	8		●									●							
	p_z	9			●												●			
	m_x	10		●		●														
	m_y	11	●				●													
C	v_x	12	●						●	●										
	v_y	13		●									●	●						
	p_z	14			●												●	●		
	m_x	15		●		●							●							
	m_y	16	●				●		●											
D	v_x	17	●						●	●	●									
	v_y	18		●									●	●	●					
	p_z	19			●												●	●	●	
	m_x	20		●		●							●	●						
	m_y	21	●				●		●	●										
E	v_x	22	●						●	●	●	●								
	v_y	23		●									●	●	●	●				
	p_z	24			●												●	●	●	●
	m_x	25		●		●							●	●	●					
	m_y	26	●				●		●	●	●									



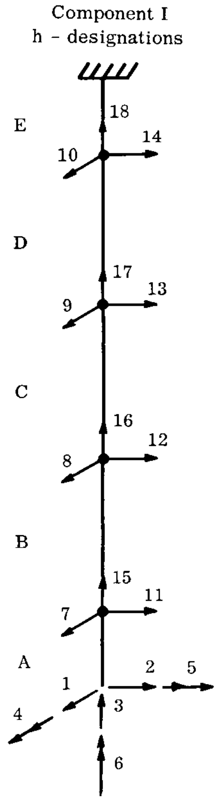


Figure C-8. $[\alpha_{mh}]$ Matrix Outline for Component I

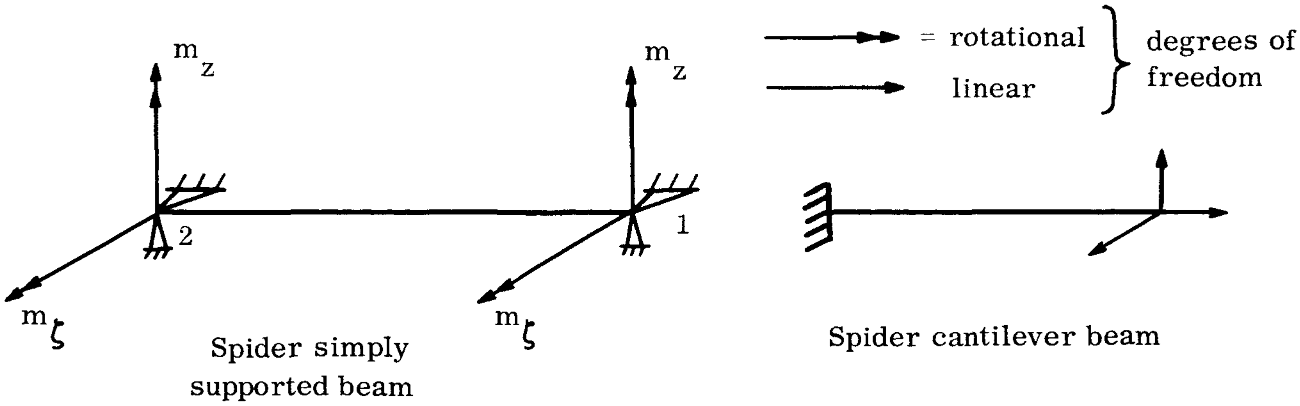


Figure C-9. Spider Beam Degrees of Freedom

The internal forces in the above discussions are determined for loads acting at any of the three types of points on the component: (1) h_i, (2) h_b and (3) h_t.

(1) h_i is the designation of the applied loads acting at the mass points of the component.

(2) h_b is the designation of the sense and location of determinate connecting forces from adjoining structure. On the central tank, component I (Figure C-8), they are the six forces at the lower end. On component II (Figure C-10) there are none because the determinate connection to the other structures is the datum for this component.

(3) h_t is the designation of the sense and location of redundant connecting forces from adjoining structure. For example, on component II these connecting redundants occur at the adjoining spiders.

Example:

Consider the $[\alpha_{mh}]$ matrix of component II, Figure C-10. Apply a unit h_i (external) force at the second mass point from the bottom in the y-direction. This is given the number 6. The reaction at the bottom is -.625 and this is the v_y on element "A". Then, at element "C", the shear is $v_y = -.625 + 1.0 = .375$ and the moment $m_x = 30(-.625) = -18.75$, pointing in the negative x-direction. When a unit redundant moment h_t is applied at the top connection (#16) (refer also to Figure C-7), a reaction of $v_x = .0125$ is caused at the bottom connection. The internal forces in element "D" are $v_x = .0125$, $m_y = -50(.0125) = -.625$ which is negative, since the moment vector points in the negative y-direction.

Element	Force type	m \ h	h_i												h_t			
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	v_x	1	•	•	•	•										•		•
	v_y	2					•	•	•	•								
	p_z	3									•	•	•	•	•			
	m_y	4														•		
	m_z	5															•	
B	v_x	6	•	•	•	•												
	v_y	7					•	•	•	•								
	p_z	8										•	•	•	•			
	m_x	9					•	•	•	•								
	m_y	10	•	•	•	•										•		•
C	v_x	11	•	•	•	•										•		•
	v_y	12					•	•	•	•								
	p_z	13													•			
	m_x	14					•	•	•	•								
	m_y	15	•	•	•	•										•		•
D	v_x	16	•	•	•	•										•		•
	v_y	17					•	•	•	•								
	p_z	18													•			
	m_x	19					•	•	•	•								
	m_y	20	•	•	•	•										•		•
E	v_x	21	•	•	•	•										•		•
	v_y	22					•	•	•	•								
	p_z	23													•			
	m_y	24																•

$[\alpha_{mh}]$ matrix outline for Component II

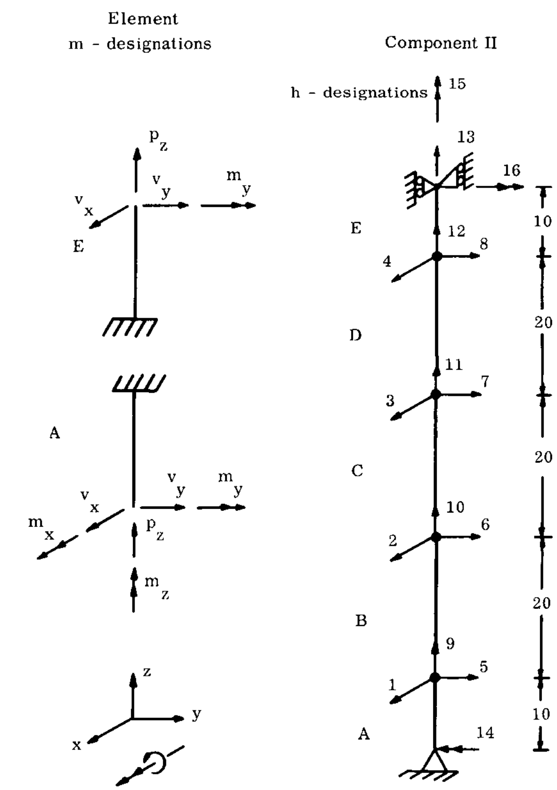


Figure C-10. Component II $[\alpha_{mh}]$ Matrix and Construction

2. $[\gamma]$ Matrix

$[\gamma]$ is the flexibility matrix of the elements. It is shown diagramatically in Figure C-11 for component I and it is of this form for all components. The matrices on the diagonal are the flexibility matrices of the elements, A, B, C, etc., of the component.

$$\left[\begin{array}{c|c|c|c|c} [\gamma_A]_{(6x6)} & & & & \\ \hline & [\gamma_B]_{(5x5)} & & & \\ \hline & & [\gamma_C]_{(5x5)} & & \\ \hline & & & [\gamma_D]_{(5x5)} & \\ \hline & & & & [\gamma_E]_{(5x5)} \end{array}\right]$$

Figure C-11. Flexibility Matrix of Component I

It should be noted that the column order is the same as the row order in the $[\alpha]$ matrix (Figure C-8). A typical matrix for a cantilever beam is shown in Figure C-12.

$$\begin{array}{c|c} & \begin{array}{cccccc} v_x & v_y & p_z & m_x & m_y & m_z \end{array} \\ \hline \begin{array}{c} \Delta_x \\ \Delta_y \\ \Delta_z \\ \theta_x \\ \theta_y \\ \theta_z \end{array} & \begin{bmatrix} \frac{\ell^3}{3EI_y} & & & & \frac{-\ell^2}{2EI_y} & \\ & \frac{\ell^3}{3EI_x} & & \frac{\ell^2}{2EI_x} & & \\ & & \frac{\ell}{AE} & & & \\ & \frac{\ell^2}{2EI_x} & & \frac{\ell}{EI_x} & & \\ \frac{-\ell^2}{2EI_y} & & & & \frac{\ell}{EI_y} & \\ & & & & & \frac{\ell}{GJ} \end{bmatrix} \end{array}$$

Figure C-12. Flexibility Matrix of a Cantilever Beam with Constant Cross-Section Properties

The signs of the matrix elements are appropriate for the structural elements oriented as on component I with respect to the reference axes. The row and column pertaining to twist is deleted for all elements except A. This element of the flexibility matrix for A was obtained by taking ℓ equal to the total length of the component.

For the spider cantilevers, only the rows and columns containing the shears and axial load are applicable with due regard for signs.

The $[\gamma]$ matrix for the middle portions of the spider beams is as shown in Figure C-13. These are simply supported beams (refer also Figure C-9) with constant cross-section properties.

$$
\begin{array}{c|c}
 & \begin{array}{cccc} m_{\zeta_1} & m_{z_1} & m_{\zeta_2} & m_{z_2} \end{array} \\
\hline
\begin{array}{c} \theta_{\zeta_1} \\ \theta_{z_1} \\ \theta_{\zeta_2} \\ \theta_{z_2} \end{array} &
\begin{bmatrix}
\frac{\ell}{3EI_\zeta} & & \frac{-\ell}{6EI_\zeta} & \\
 & \frac{\ell}{3EI_z} & & \frac{-\ell}{6EI_z} \\
\frac{-\ell}{6EI_\zeta} & & \frac{\ell}{3EI_\zeta} & \\
 & \frac{-\ell}{6EI_z} & & \frac{\ell}{3EI_z}
\end{bmatrix}
\end{array}
$$

Figure C-13. Flexibility Matrix of a Simply Supported Beam

3. $[\beta]$ Matrix

The only components having internal redundants are the spiders, for which these matrices exist. The row designations are identical to those of the $[\alpha]$ matrix. Each of the columns gives the internal forces in the elements caused by a particular unit redundant force. The elements are computed similarly to those of $[\alpha]$, but the equal and opposite forces on a cut which constitute a redundant are considered simultaneously.

4. Coupling of Components

a. [a] Matrix

Consider the components connected in a statically determinate manner. A force influence matrix is formed from the application of unit external loads. Each element in a column is equal to the external force sustained by a point due to a load applied at that point or another point on the structure. In this problem there are no loads applied externally at redundancies between components, so that there are no columns corresponding to t_p points. For each component there are a number of external load points and connection points, h, for which the values of the forces are sought. The h rows of [a] correspond exactly to the columns of the [α] matrix. The rows are classified as h_i, h_b, h_t. As there are no external loads at t_p points, all entries on the rows corresponding to the redundant connection points (h_t) are zero. The external load rows (h_i) have entries only when the load is on that specific point of the component and are therefore on the diagonal of the matrix. When loads are applied to other components there will be forces at the determinate connection points (h_b) between the component and the other structure.

An outline of the [a_{hj}] matrix for component I is given in Figure C-14. Refer to Figure C-2 for the j-designations and to Figure C-8 for the h-designations. When a unit load P_j at j=2 is applied on the component itself, it will only cause a force of 1 at the point h_j=8. When a unit load P_j is applied on another component, say at j=8, there will be forces at the connection of component VI to component I. In this case there is a positive shear in the x-direction of .125 and a positive torque of 9.5(.125)=1.1875 which are the specific values of the elements checked for the points h_b = 1 and h_b = 6 for component I.

b. [b] Matrix

The components are connected in a statically determinate manner as for the [a] matrices. The values for the rows of the [b] matrix are obtained by applying unit redundant force pairs, which occur at the connections between the components, and calculating the forces at the h-points of each component. The row labels are therefore the same as for the [a] matrix. The

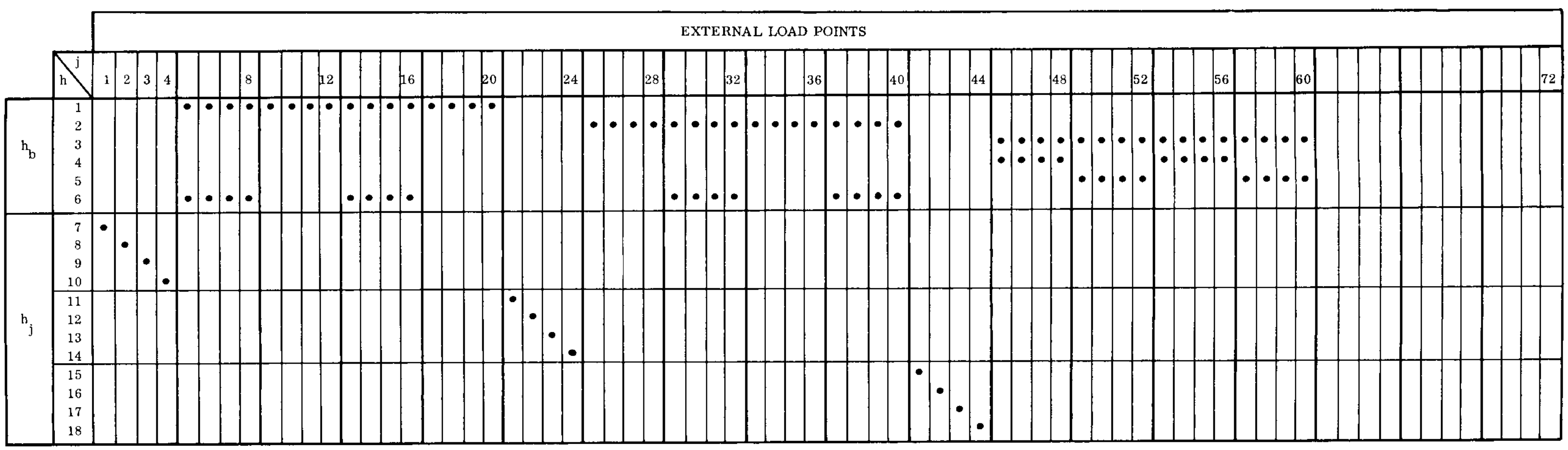


Figure C-14. Outline of $[\alpha_{hj}]$ Matrix Elements for Component I numbering and ordering of rows is the same as that of the columns of the $[\alpha]$ matrix of each component. The outline of these matrices for components I and II is given in Figure C-15.

D. THE FLEXIBILITY MATRIX

The flexibility matrix $[\gamma_{ij}]$ of the sample problem was obtained by using the method of this report, and the specific data generated by the methods of the above outline. It was calculated by using Republic Aviation digital program 64D017 on the IBM 7090 digital computer.

Those flexibilities which could be calculated exactly by hand were checked against the computed values. The matrix elements were also checked for symmetry and antisymmetry of displacements corresponding to certain unit applied loads. Those elements which were expected to have zero values were sufficiently close to zero, if not absolutely so, when compared to the diagonal elements to be interpreted as such. All checks were satisfactory. Therefore, it is concluded that this matrix is the correct flexibility matrix for the subject structure.

Coupling Redundant Points

	h \ t	1	2	3	4	5	6	7	8	9	10	11	12
h_b	1					●	●	●	●				
	2			●	●								
	3	●	●										
	4	●	●	●	●								
	5					●		●					
	6			●	●	●	●	●	●	●	●	●	●
h_j	7												
	8												
	9												
	10												
	11												
	12												
	13												
	14												
	15												
	16												
	17												
	18												

Outline of $\left[b_{ht} \right]$ matrix for Component I

Coupling Redundant Points

	h \ t	1	2	3	4	5	6	7	8	9	10	11	12
h_j	1												
	2												
	3												
	4												
	5												
	6												
	7												
	8												
	9												
	10												
	11												
	12												
h_t	13	●											
	14					●							
	15									●			
	16						●						

Outline of $\left[b_{ht} \right]$ matrix for Component II

Figure C-15. $[b_{ht}]$ Matrices

SECTION III - CALCULATION OF NATURAL MODES OF VIBRATION

A. DETERMINATION OF THE DYNAMIC MATRIX

1. Mass Matrix

The mass matrix was obtained by assuming all the mass to be in the tanks. The spider beams were considered as massless springs.

The length, diameter, and thickness of the tanks were scaled from consideration of the Saturn vehicle. The tanks were assumed to be completely filled with water. A sketch of the model is shown in Figure C-2. Four mass points per tank were used to describe the system for the sample problem. Each of these mass points were given three translational degrees of freedom. Table 10 shows the masses assigned to each mass point.

2. Transformation Matrix for the "Free-Free" Dynamic Matrix

The influence coefficient matrix obtained in the previous section was referenced to a plane connecting the four support points of the central tanks to the spider. It is thus held at this plane. To ascertain the natural vibration modes of the vehicle in flight, it is necessary to release or "free-free" the vehicle. This is done in the following manner. The equations of motion of the unrestrained or "free-free" system can be written as:

$$\left\{ y_i - y_0 - x_i \theta_0 \right\} = \omega^2 [C] [M] \{y_i\} \qquad \text{(C.1)}$$

where

$[C]$	=	influence coefficient matrix for the restrained vehicle
$[M]$	=	mass matrix
y_i	=	deflection of ith station
y_0, θ_0	=	deflection and rotation of station previously considered as support point
x_i	=	distance from support point station to ith station
ω	=	natural vibration frequency (rad/sec)

TABLE 10

NATURAL VIBRATION MODES SAMPLE PROBLEM MASS DATA

Degree of Freedom			Mass
x	y	z	
1	21	41	.15165
2	22	42	.15165
3	23	43	.15165
4	24	44	.15165
5	25	45	.07554
6	26	46	.07554
7	27	47	.07554
8	28	48	.07554
9	29	49	.07611
10	30	50	.07611
11	31	51	.07611
12	32	52	.07611
13	33	53	.07554
14	34	54	.07554
15	35	55	.07554
16	36	56	.07554
17	37	57	.07611
18	38	58	.07611
19	39	59	.07611
20	40	60	.07611
61	65	69	.15165
62	66	70	.15165
63	67	71	.15165
64	68	72	.15165

In addition, for a three-dimensional system there are six equilibrium equations of the form

$$\sum_{i=0}^{n} m_i y_i = 0 \tag{C.2}$$

$$\sum_{i=0}^{n} m_i x_i y_i + I_0 \theta_0 = 0 \tag{C.3}$$

Equations (C. 2) and (C. 3) are then used to eliminate y_0 and θ_0 from Eq. (C. 1). The transformation matrix [F] then has the form

$$[F] = \left[[I] - [B]^T [G]^{-1} [B] [M] \right] \tag{C.4}$$

where

$[I]$ = unit matrix n x n

$[B]$ = matrix of rigid body modes, i. e., translation, rotation, etc.

$$[G] = \begin{bmatrix} M_1 & S_1 & & & & \\ S_1 & I_1 & & & & \\ & & M_2 & S_2 & & \\ & & S_2 & I_2 & & \\ & & & & M_3 & S_3 \\ & & & & S_3 & I_3 \end{bmatrix}$$

where M_i = total mass in ith direction

S_i = unbalance about reference axis

I_i = moment of inertia about reference axis

$[M]$ = mass matrix

So that the final dynamic matrix is

$$\{y_i\} = \omega^2 \left[[F] [C] [M] \right] \{y_i\} \tag{C.5}$$

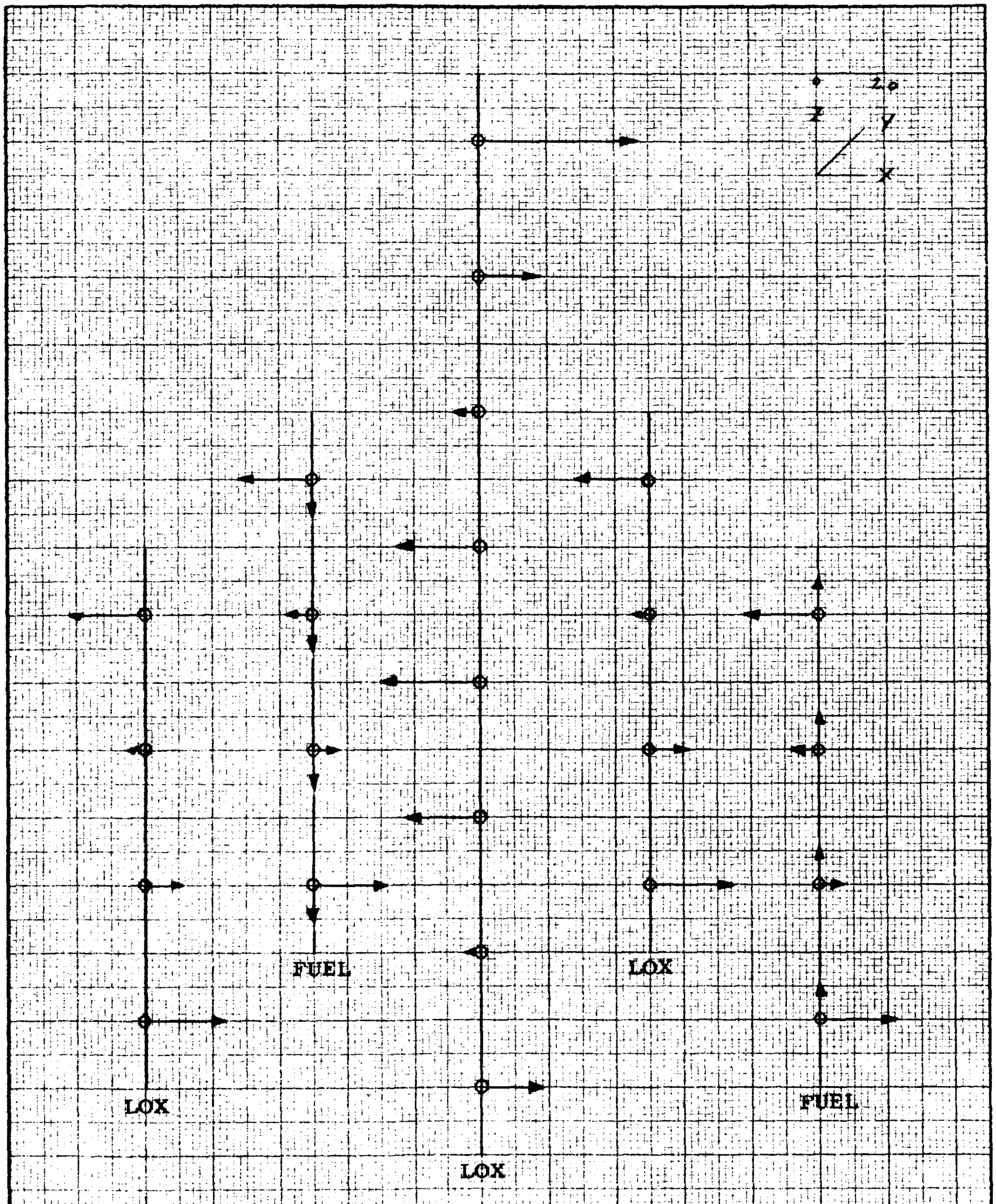
and this is the matrix which is operated upon to obtain the "free-free" eigenvalues and eigenvectors, corresponding to the natural frequencies and modes of the vehicle in flight.

B. DESCRIPTION OF THE NATURAL VIBRATION MODES

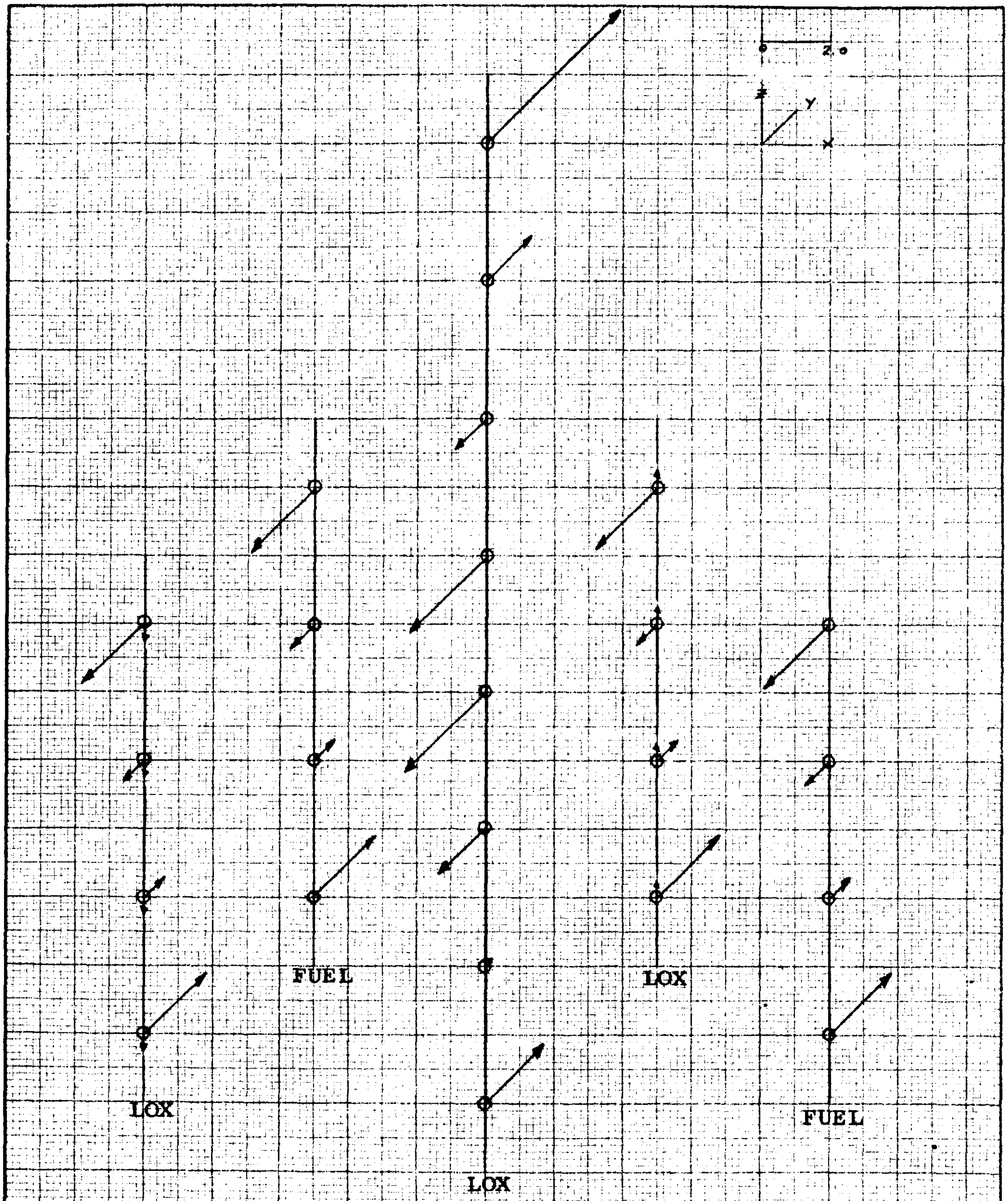
The configuration described in the previous section has been studied for its natural modes of vibration. In applying the Lanczos method of minimized iterations, it must be borne in mind that any mode which is orthogonal to the initial starting vector will not be obtained. To guard against this eventuality, a starting vector was selected which, it was believed, did not represent a conceivable mode of the structure, and hence would not be orthogonal to the other modes of the structure. As a further precaution, two such starting vectors were employed. The computer was instructed to proceed with the iteration until the first seven eigenvalues had converged to five significant figures each. For both initial starting vectors this convergence was recognized in 15 iterations, i.e., n = 16.

For this sample problem only the first seven modes of the structure have been obtained. For the purpose of describing the deflections, the deformation is defined in terms of the three orthogonal translations x, y, and z. Here x represents deflections perpendicular to the plane formed by the center lines of the two peripheral lox tanks (see Figure C-2); y represents lateral deflections perpendicular to x; and z represents vertical motion.

The two lowest modes of the vehicle (Figures C-16 and C-17) exhibit the characteristics of basic first free-free bending of a beam in the two principal transverse directions. The different spider attachment conditions for the two types of peripheral tanks cause these two first bending modes to have different natural frequencies. Inasmuch as the connection of the fuel tanks to the upper spider is more flexible than that of the lox tanks, the principal axis in the lowest bending mode is the axis formed by connecting the three lox tanks (see Figure C-1). The second mode (Figure C-17) has its principal axis perpendicular to this. It is seen that the peripheral tanks off the principal axis translate vertically out of phase with each other.



1st Mode
$f_n = 12.502$ cps
Figure C-16



2nd Mode
$f_n = 14.512$ cps

Figure C-17

The third mode (Figure C-18) involves a torsion-type deformation pattern. There is no vertical deflection, the motion being essentially a twisting of the longitudinal axis.

The fourth and fifth modes (Figures C-19 and C-20) represent primarily bending of the peripheral tanks in the two transverse directions. The lower of these two modes involves large motions of the fuel tanks in the x-directions. These are opposed by smaller motions of the lox tanks of the first stage out of phase with the fuel tanks. In the fifth mode the x-motion is replaced by y-motion. Inasmuch as no mass points were assumed for the spider, the extent to which spider flexure is present in these modes cannot be readily determined. These modes fall under the heading of cluster-type modes.

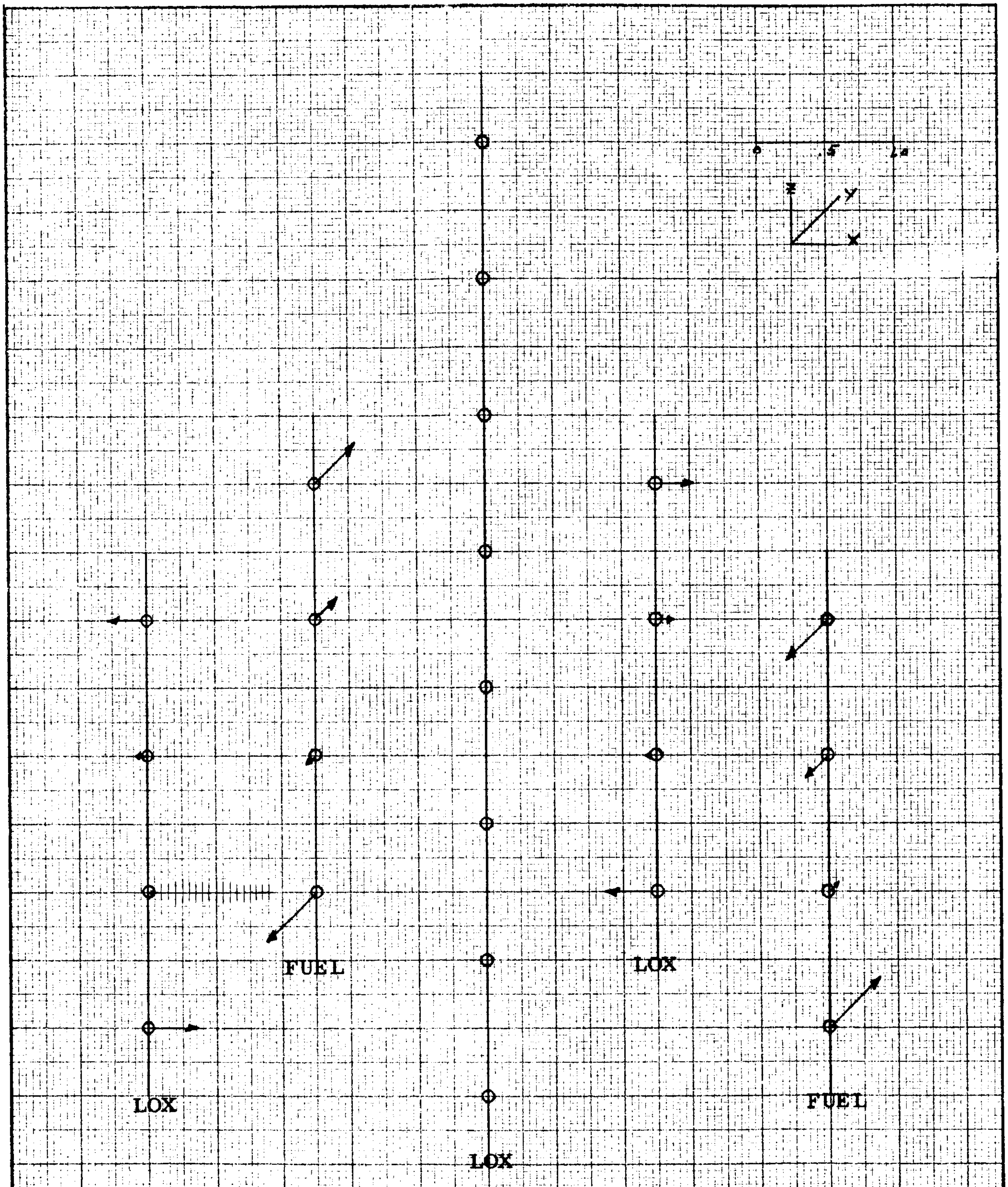
The sixth mode (Figure C-21) is another cluster-type mode. This mode exhibits the characteristic of the peripheral tanks moving in the x-direction out of phase with the central tank.

The seventh mode (Figure C-22) is also a cluster-type mode and appears to be somewhat similar to the sixth mode rotated 90 degrees, with one significant difference. In the fifth mode, all the peripheral tanks are in phase; in the seventh mode, this condition no longer holds. It is true, however, that both these modes involve lox tank motion primarily.

All cluster-type tank modes exhibit somewhat of a rocking motion to some extent, wherein opposite peripheral tanks will be translating vertically out of phase with each other.

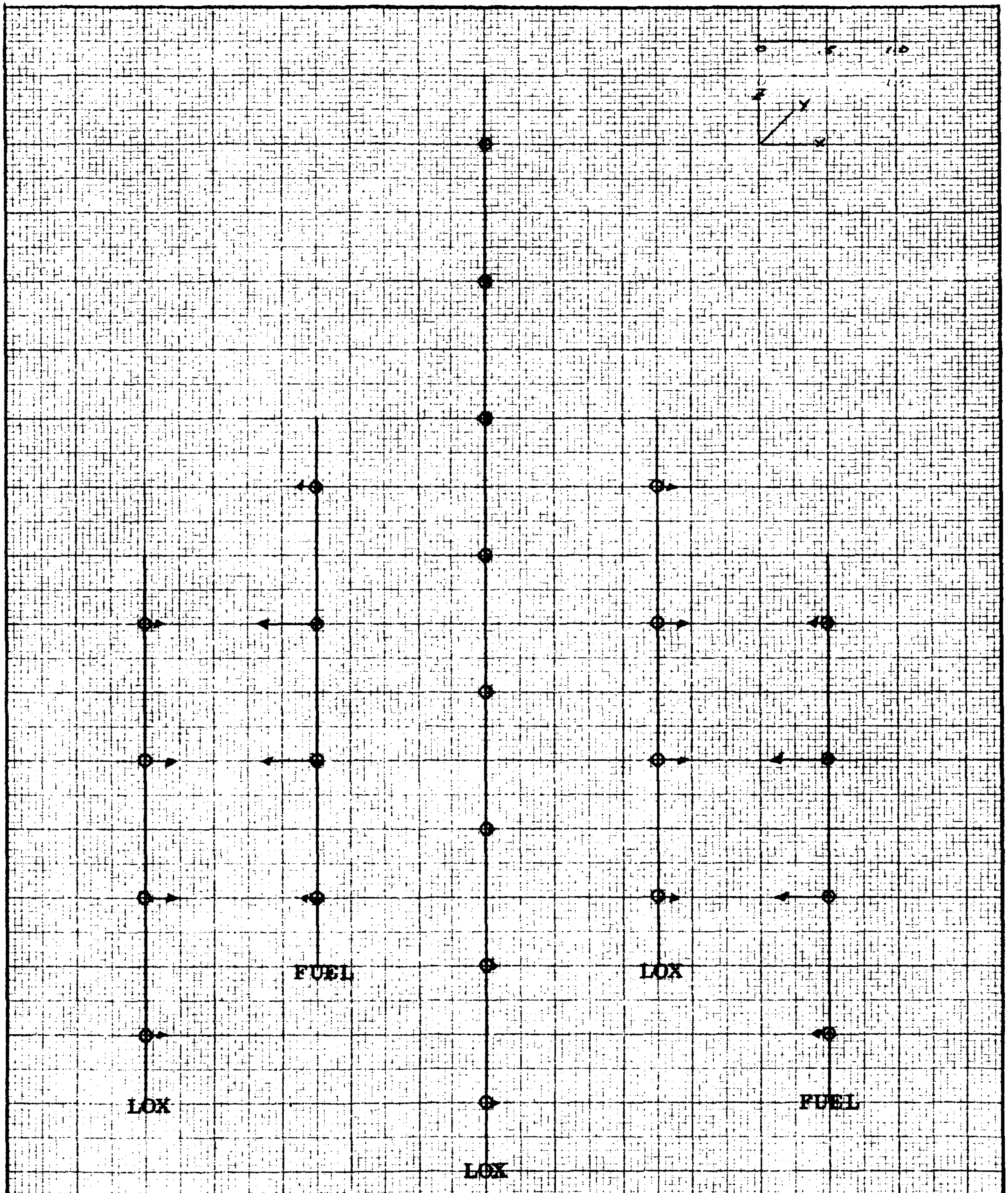
The seven modal frequencies are calculated as:

1. 12.502 c.p.s.
2. 14.512 c.p.s.
3. 19.075 c.p.s.
4. 23.210 c.p.s.
5. 24.446 c.p.s.
6. 27.934 c.p.s.
7. 28.526 c.p.s.



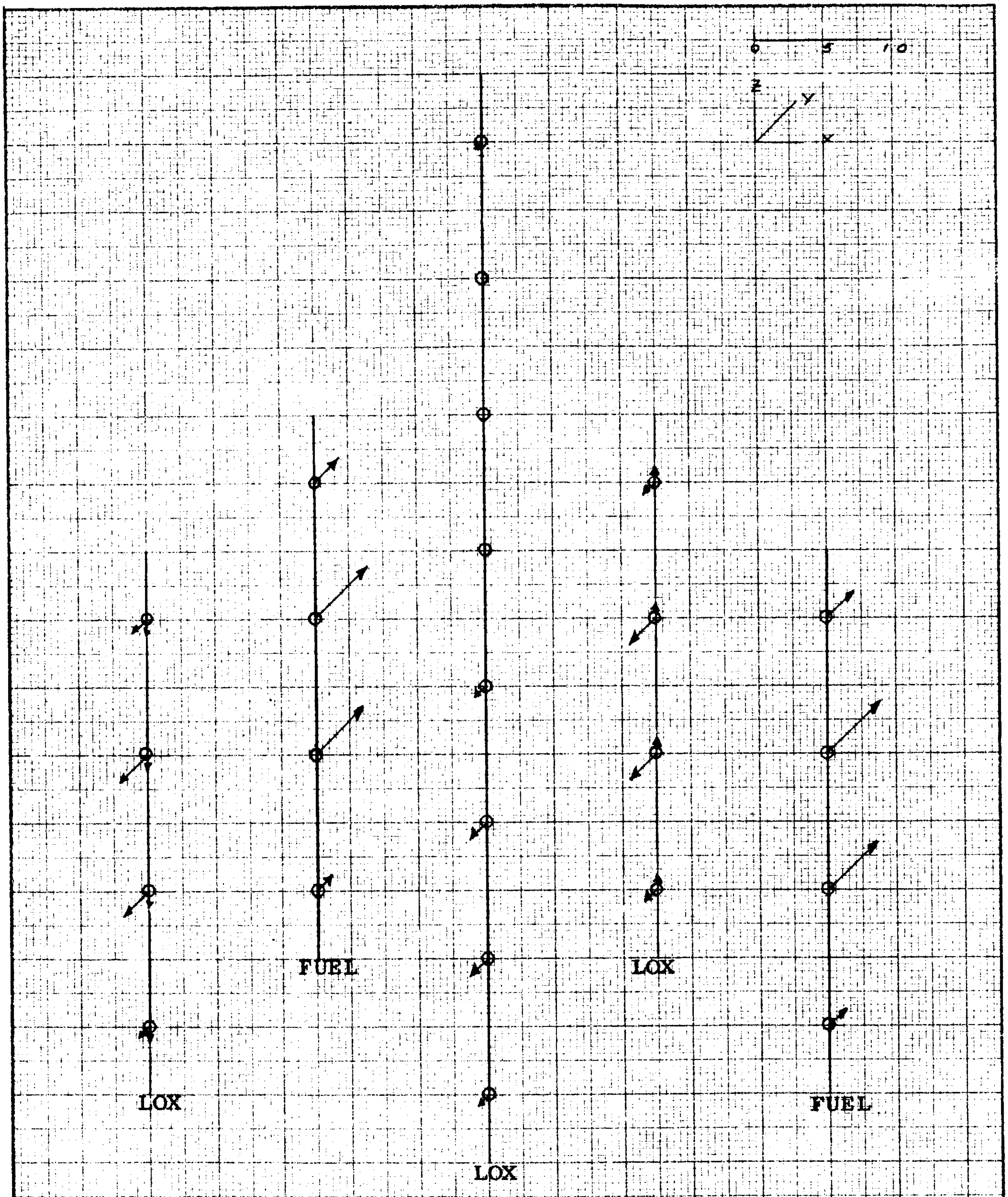
3rd Mode
f_n = 19,075 cps

Figure C-18



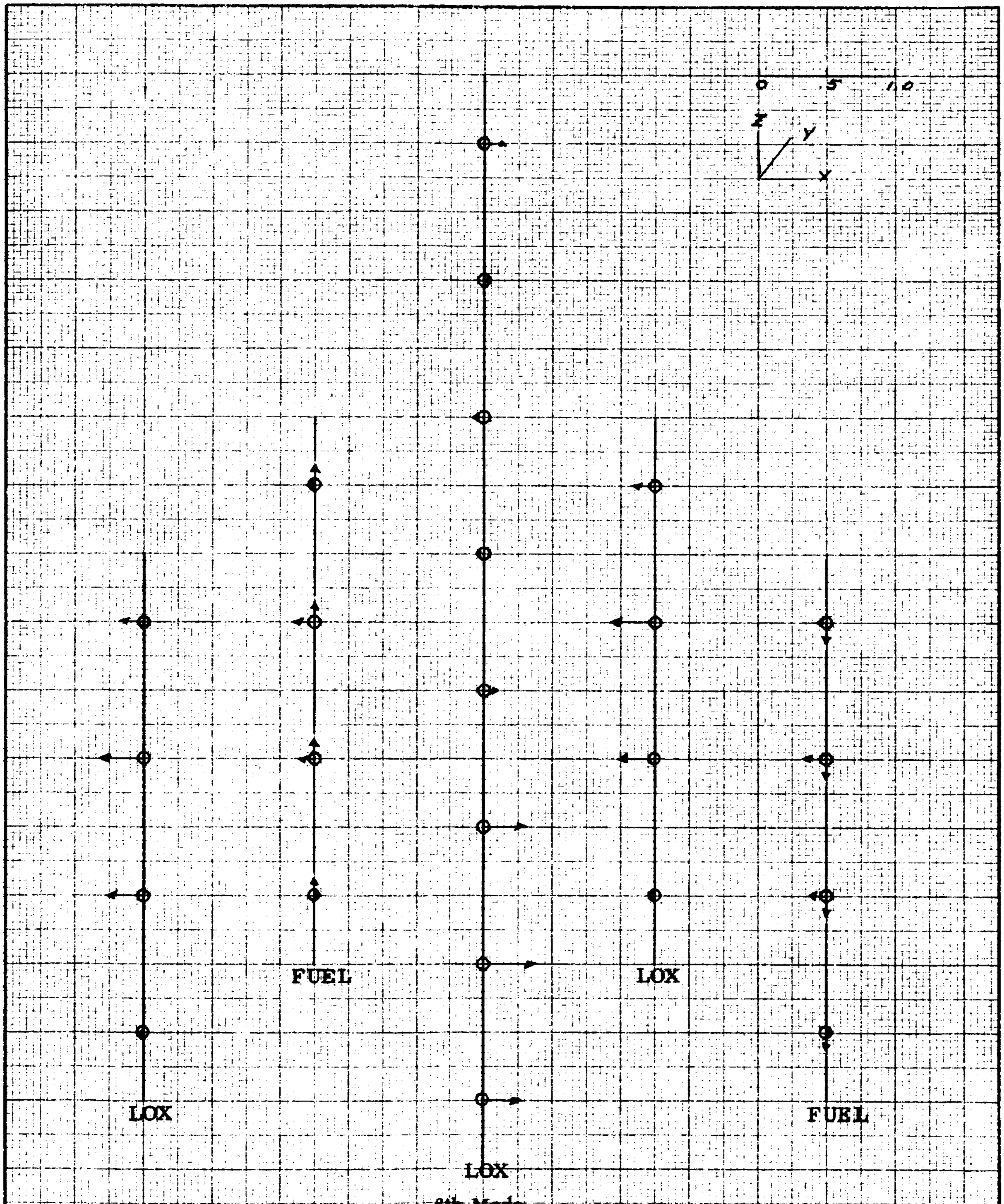
4th Mode
f_n = 23.210 cps

Figure C-19



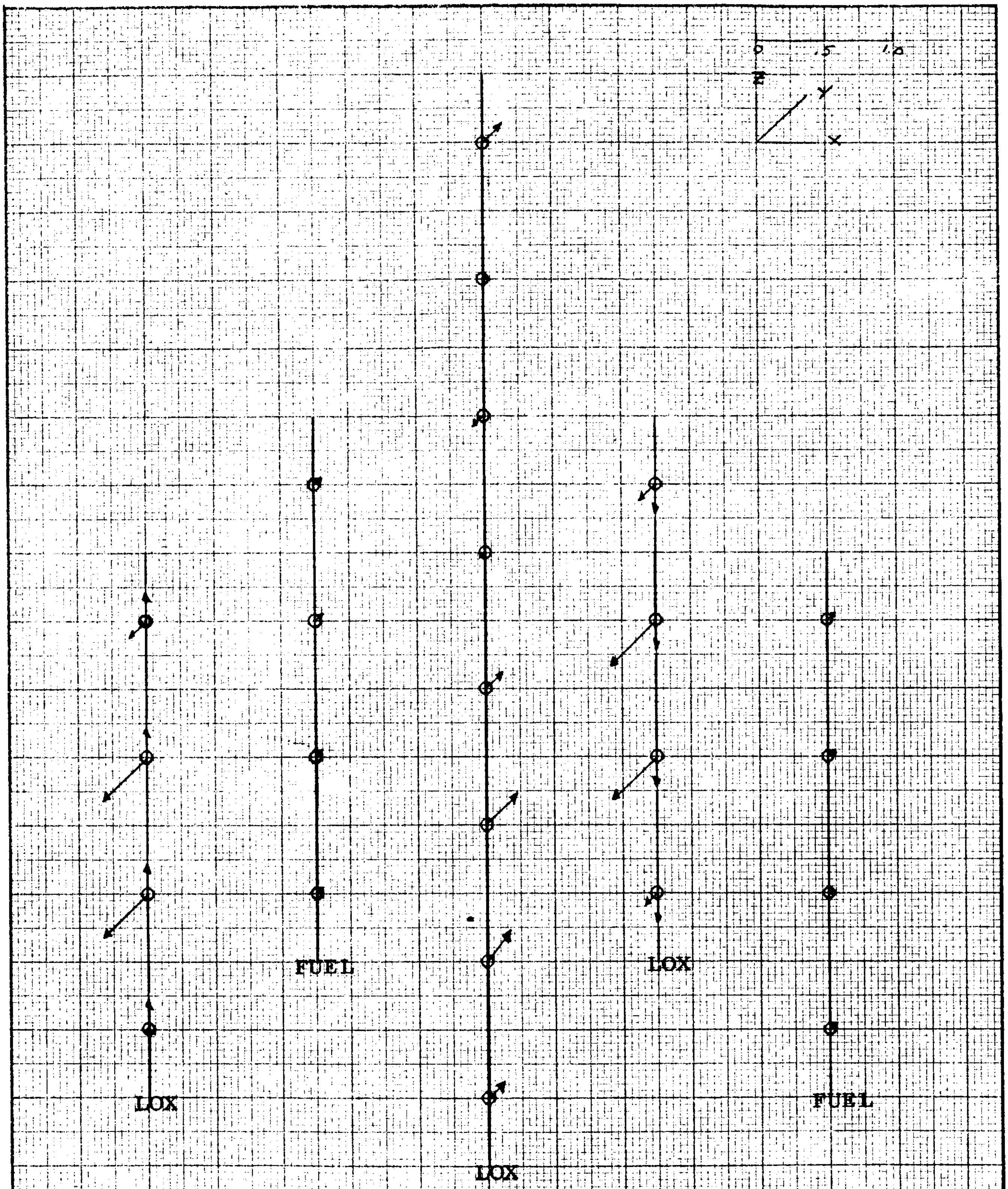
5th Mode
$f_n = 24.446$ cps

Figure C-20



6th Mode
f_n = 27.934 cps

Figure C-21



7th Mode
$f_n = 28.526$ cps

Figure C-22

It should be noted that the cluster-type modes are bunched rather closely. This is undoubtedly due to the essential symmetry inherent in the system, and the small decoupling effect introduced by the different end-fixity conditions and the somewhat different masses for the fuel and lox tanks (see Table 10). This bears out the ability of the procedure to detect closely bunched natural frequencies.

C. COMMENTS FOR EFFICIENT APPLICATION OF THE METHOD OF MINIMIZED ITERATIONS

The calculation consisted of two phases. The first phase is the determination of the eigenvalues. At each iteration the roots of the resulting polynomial are derived. The iteration proceeds until the desired number of roots have converged. The following table shows the first seven roots of the last three polynomials. The roots were obtained using triple precision, but only the first seven significant figures are employed in the final print-out. The test for convergence involved the first five significant figures.

	XN_{16}	XN_{15}	XN_{14}
1.	3.241388	3.241388	3.241388
2.	2.405435	2.405435	2.405435
3.	1.392283	1.392166	1.289137
4.	.9404117	.9404117	.9403751
5.	.8477335	.8477335	.8477320
6.	.6492513	.6492510	.6492006
7.	.6225753	.6225750	.6225239

It is interesting to note that the third mode apparently was the last of the above to converge. This is undoubtedly due to the fact that the scalar product of the initial starting vector and the vector representing the third mode is comparatively small. If one considers the initial starting vector as a kind of forcing function, the relatively slow convergence of the third eigenvalue can be attributed to the fact that the starting vector used is a poor forcing function for this particular mode. Nevertheless, it can be seen that the procedure has sensed this mode and determined it. This makes it apparent, however, that the proper application of this method requires a familiarity on the part of the user with the anticipated nature of the modal patterns. A judicious choice of starting vectors will result in efficient and rapid determination of the desired modes and frequencies with a minimum of starting vectors and machine time. Inasmuch as this phase of the program as presently constituted required 60–75 minutes of machine time for each starting vector, rapidity of convergence effects a significant saving in computer time.

Initially it was contemplated that the machine would enter the eigenvector phase of the program directly upon realizing that the desired number of eigenvalues had converged. It was found, however, that the large number of computer tapes required to accomplish both the eigenvalue and eigenvector phase for the sample problem introduced many machine problems. For this reason the eigenvector phase of the program has been reprogrammed as a separate program which employs as input the results of the eigenvalue phase. The eigenvector phase required three minutes for the sample problem. The program employs Eq. B. 22 for the generation of the eigenvalues. Inasmuch as this requires the evaluation of the various polynomials at the root, it is apparent that, as soon as convergence is attained for a particular eigenvalue, the series can be truncated. It has been found that, if the series is not truncated at this point, numerical "hash" due to round-off error may accumulate and may eventually obscure the resulting eigenvector. The procedure as presently constituted can print out the cumulative eigenvector so that the formulating series can be truncated where desired.

SECTION IV - RECOMMENDATIONS

It is apparent that with present computer facilities there is a maximum size problem for which this method is practical. In order to determine this maximum size, a larger scale problem should be attempted. One such system is the actual Saturn vehicle which would constitute a large scale application. The results of such a study could be checked against test results as a further assessment of the usefulness of the method described in this report.

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